The Majority Game on Regular and Random Networks

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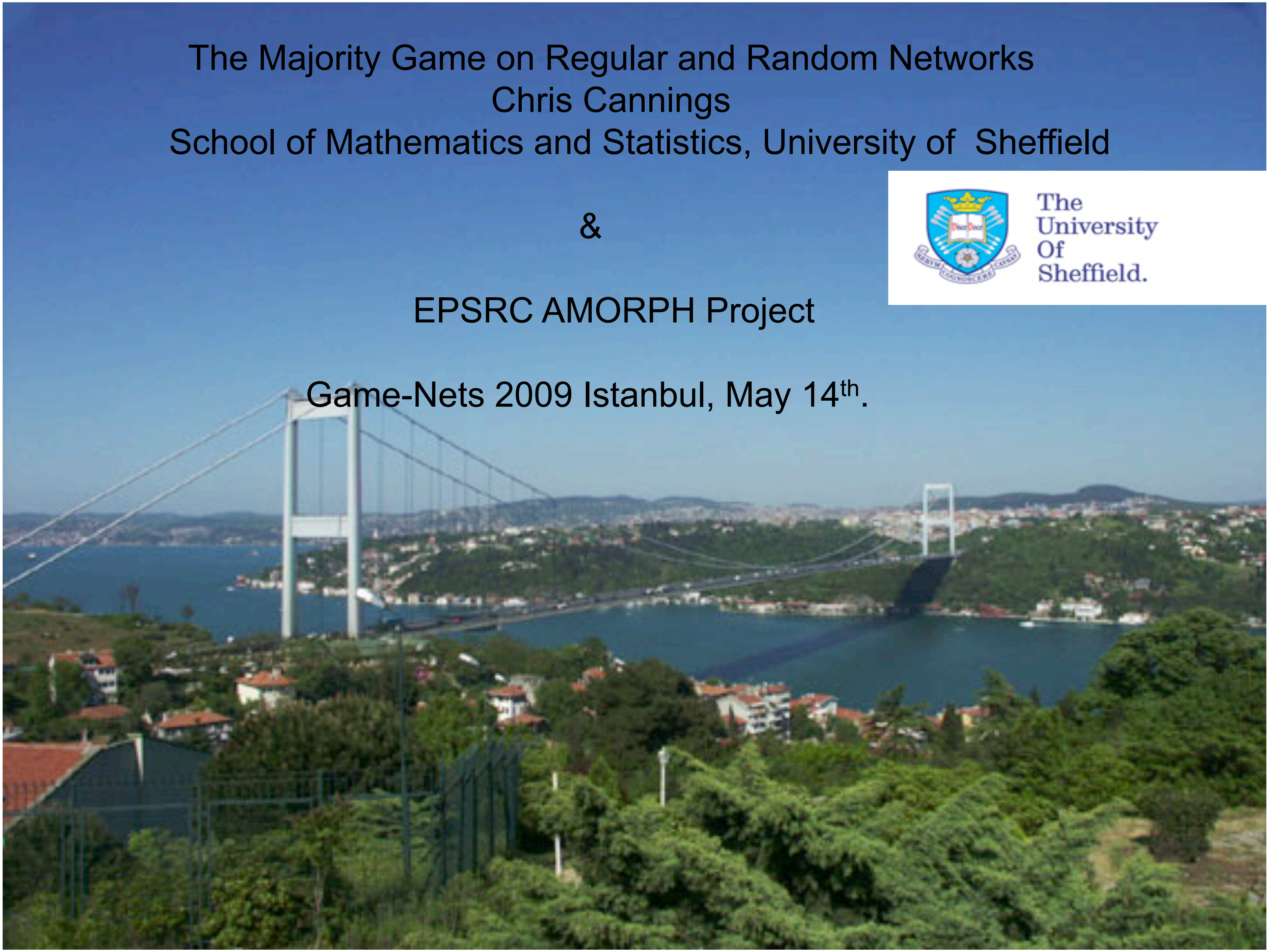

Games on Networks

- G=(V,E), V=vertex set, E=edge set.
- State of vertex , {0,1} only here.
- At time t some subset of vertices S_t update their states. A vertex, i say, in S_t looks at a subset of its neighbours, and reset its own state according to the observation, and to some rule (function).

Synchronous Games

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We consider only synchronous games. Thus each vertex plays against all of its neighbours at each time t=0,1,2,3,…

Mixed Majority/Minority Games

- The thrust of this presentation is the examination of games where individuals, located at the vertices of a graph, are either consistently conformist (they copy their neighbours), or are consistently non-conformist (they play the opposite of their neighbours).
- The conformists are Majority Players, the non-conformists are Minority Players

Majority Player

- A majority player M, is an individual who, resets (for time t+1) to state 1(0) if it observes a majority of its neighbours, at time t, as being in state 1(0).
- This is a “conformist” strategy.
- If payoff for i versus j is $a_{i,j}$ and plays against each of the neighbours viewed, this is the myopic best response strategy.

$$A = (a_{i,j}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A Minority Player

- A minority player m, is an individual who, resets (for time t+1) to state 1(0) if it observes a minority of its neighbours, at time t in state 1(0).
- This is a "non-conformist" strategy.
- If payoff for i versus j is $a_{i,j}$ and plays against each of the neighbours viewed, this is the myopic best response strategy.

$$A = (a_{i,j}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

“El Farol” Bar Problem

- The minority player was introduced by Anderson in the “El Farol” Bar scenario.
- There are two bars. An individual checks which bar others were at one evening , and goes to the other bar the next evening; to avoid the crowds. This has applications also in “trading”, where being a buyer in a market with an excess of sellers may be advantageous or choosing a route to drive to work on the basis of yesterday’s traffic.

Majority Players

- Individuals seeking a lively night-club may adopt the opposite strategy to the above and go where others went the previous evening.
- Ants following trails will choose to follow those trails which have been most reinforced by other ants. Individuals may choose a restaurant which seems to have more customers (the food must be good!).

Threshold Games

- In a threshold game each vertex v has a threshold b_v and plays 1 at time t+1 iff number of neighbours with 1 at time t exceeds b_v.
- On any graph this always leads to fixed points or to 2-cycles. Goles (1987), Berninghaus and Schwable (1996).
- Majority Game is a threshold game, as is minority game (with a simple swop of labels).

The Majority Game

- Everyone is a majority player.
- This should lead to a consensus in the population. Everyone should play 0 or everyone should play 1.
- We shall see that this need not be the outcome on graphs; indeed may be only exceptionally.

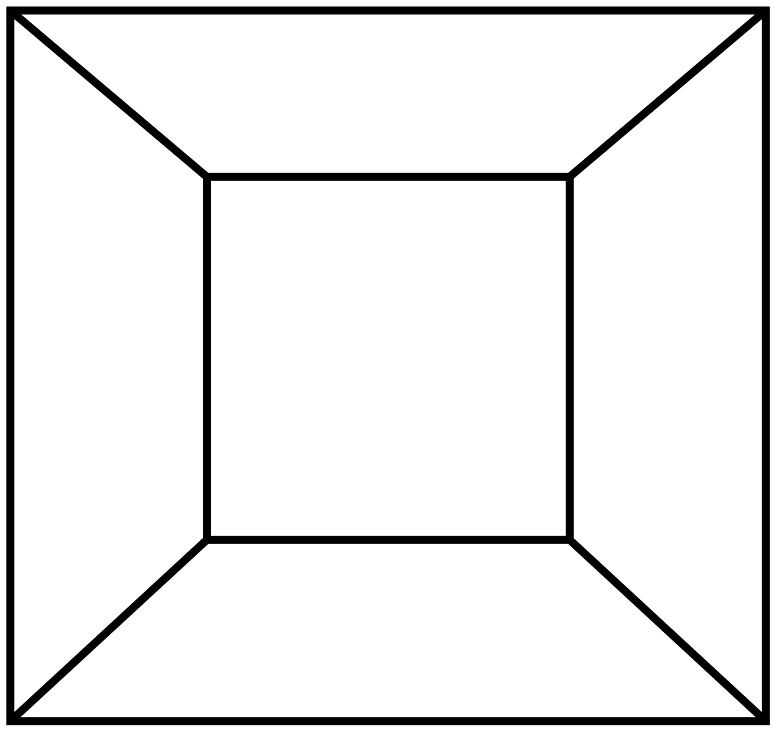
Majority Game

- Often games will be played in relatively small groups (families, clubs, market).
- Suppose that the set of individuals repeatedly play games of this type and that in each such game the initial state is chosen randomly and then the group plays a series of trials of reasonable length (so the limit is reached and used for some number of trials; relevant for payoffs).

Regular Graphs

- In an attempt to understand the behaviour of such games (and generalisations) in relation to graph parameters we consider various special regular graphs.
- Consider cubic (trivalent graphs). Every vertex has three neighbours. This avoids “draws” (i.e. neighbourhoods with equal numbers of 0’s and 1’s).

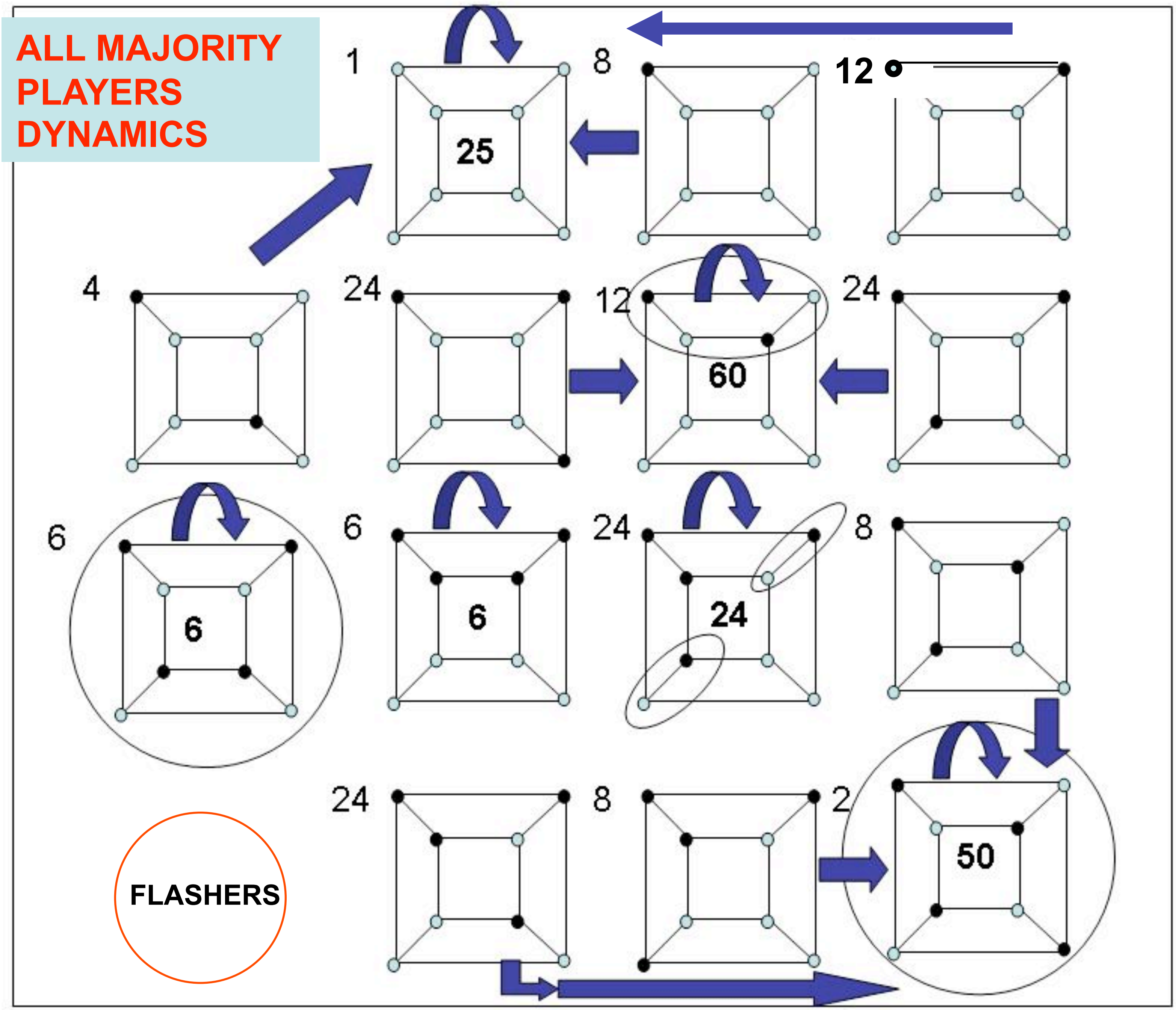
Example. 3-cube



Majority Game

- There are 256 possible states (the 0’s and 1’s) here and 22 permutationally distinct configurations (of 0’s and 1’s on a cube).

- We consider the outcomes when the initial state is assigned with equal probability (1/256).


ALL MAJORITY PLAYERS DYNAMICS
1
25
8
12
4
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12
60
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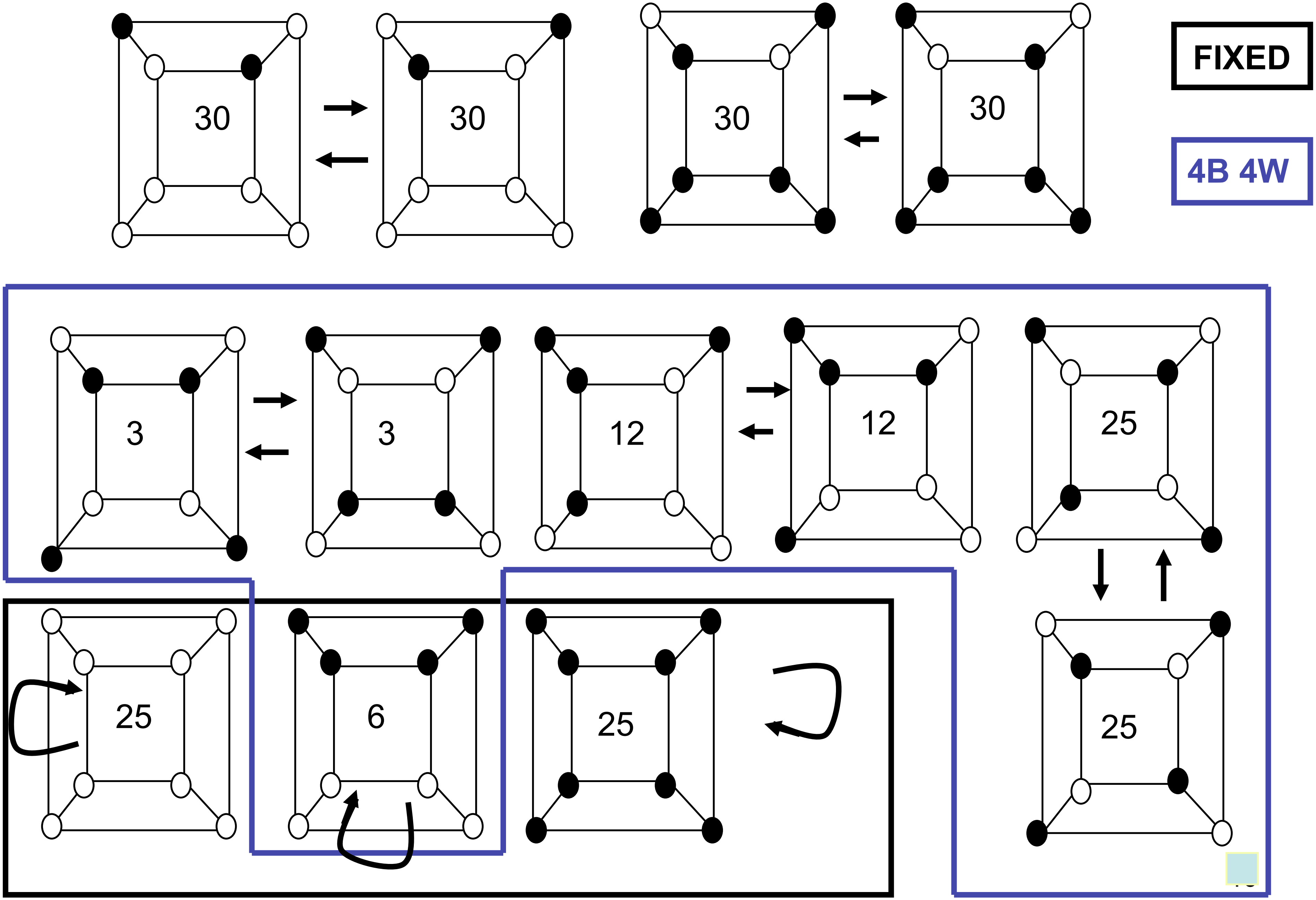
FLASHERS

LIMIT CYCLES FOR MAJORITY GAME.



Limit Cycles for Majority Game on 3-cube

- There are 3 fixed points and 5 2-cycles.
- Convergence is in 1 step.
- Probability of consensus 50/256.

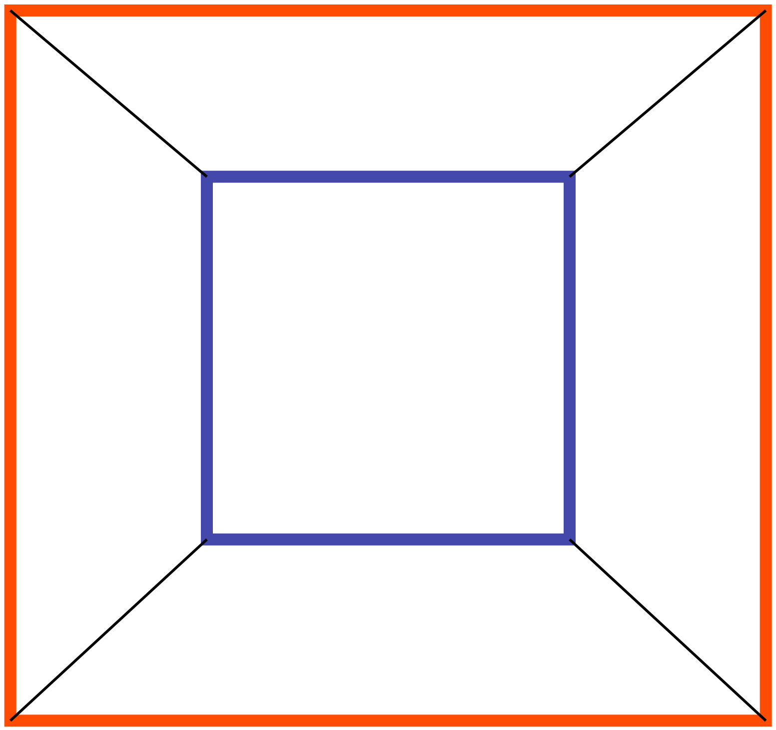
Fixed Points of the Majority Game on Cubic Graphs.

- Suppose we have a state where every vertex is fixed (0 or 1) through time.
- A vertex (playing wlog 0) must have at least two 0-neighbours. Each of these must have at least two 0-neighbours i.e. one additional to our original one, and so on.
- Thus every 0-vertex must be in a circuit of 0-vertices. Similarly for 1-vertices.

Cycle-Partitions of Cubic Graphs

- For a graph G={V,E} a cycle partition is $\{G_1, G_2, \ldots\ldots, G_k\}$ being k graphs induced by G on the elements of a partition of $V=\{V_1, V_2, \ldots\ldots, V_k\}$; such that every vertex of G_i belongs to a cycle in G_i.
- Given a cycle-partition then we can assign 0's and 1's to the elements G_i to give a fixed point of the majority game.

Cycle-Partitions of the Cube



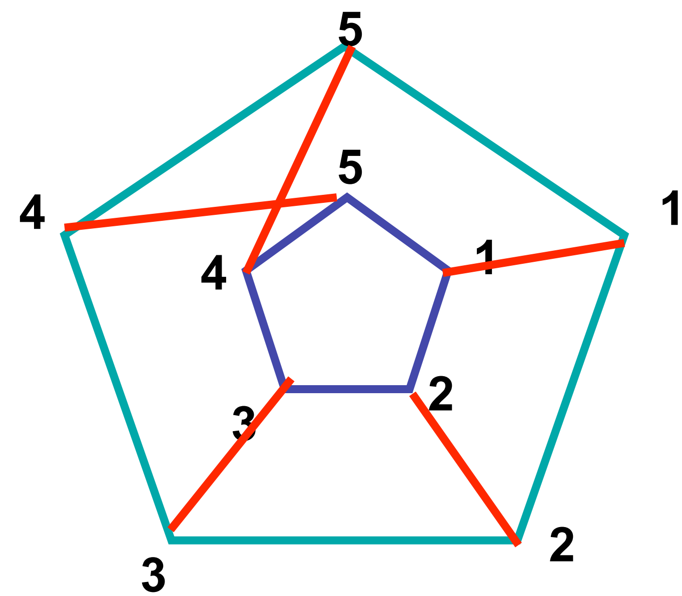
Pairs of Opposite faces. Thus 3 distinct cycle-partitions, 8 distinct fixed points.

Other Cubic Graphs

- Our examples are for n=4, 6, 8 and 10.
- There are 1, 2, 5, 19, 85,…connected cubic graphs with (resp) 4, 6, 8, 10, 12 vertices.
- A random cubic graph will generically have little, or no, symmetry. Here I choose some which have more symmetries, the order of the automorphism group of the graph..

A class of “cylindrical” cubic graphs

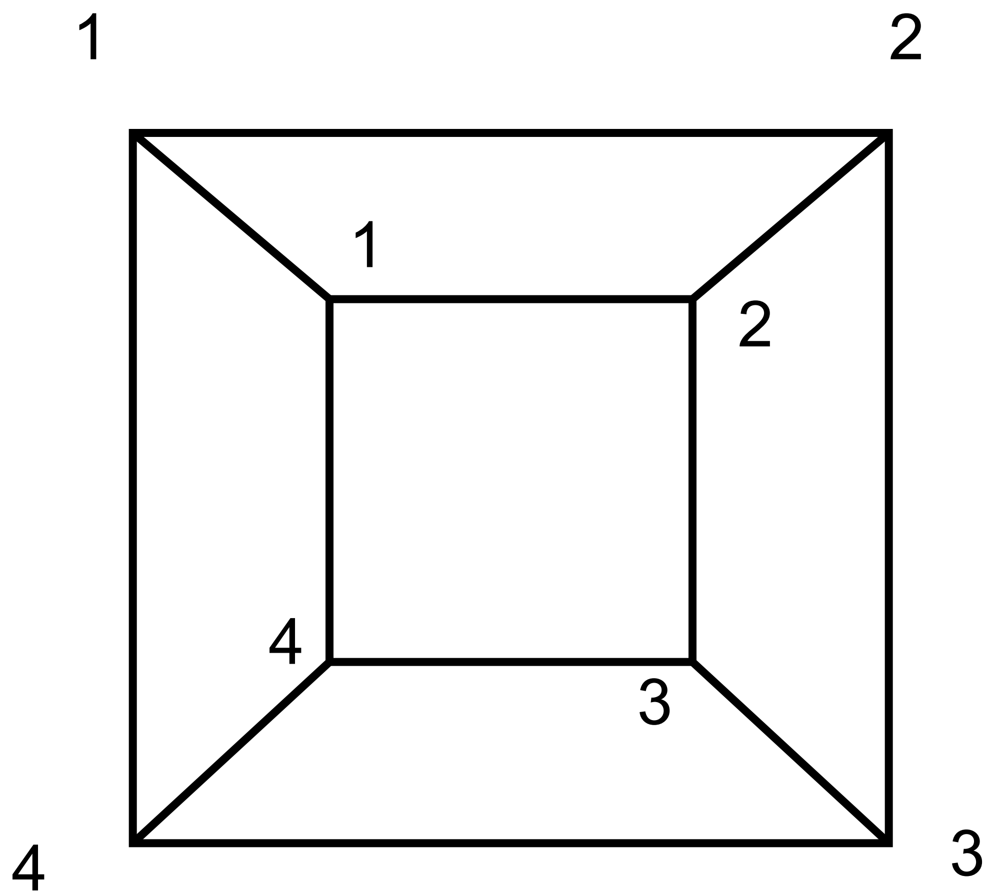
- Take two polygons of size n (here n=5).



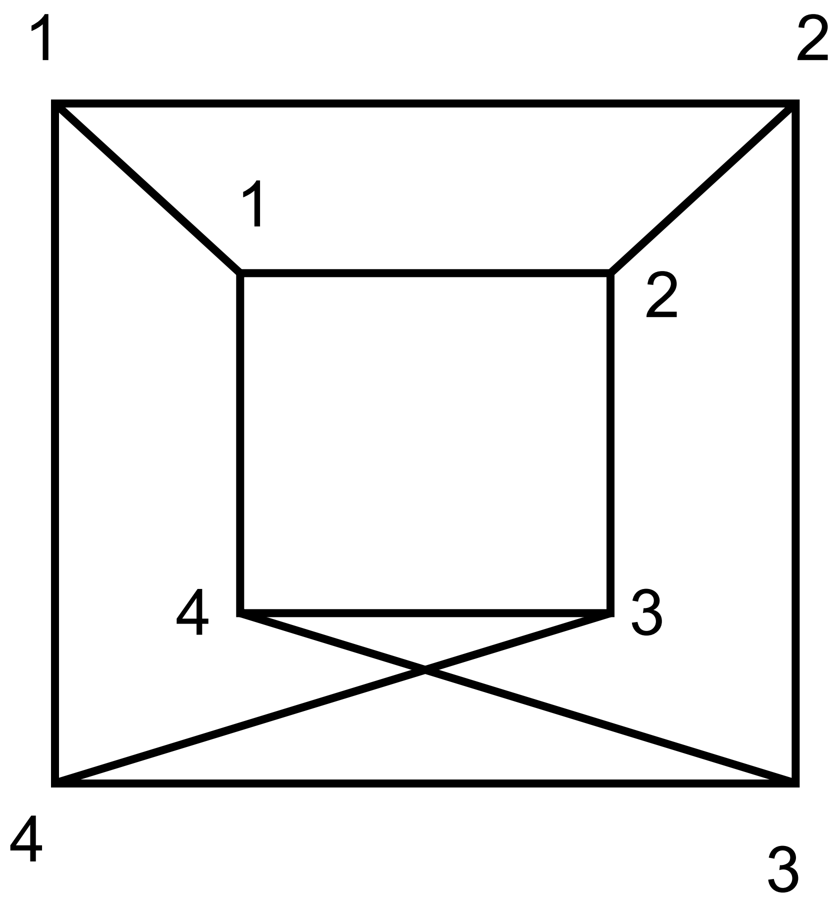
Take a permutation $P=\{p_1,p_2,\ldots\ldots\ldots\ldots p_n\}$, and join each i on one polygon to p_i on the other.
Example (shown) perm=(1,2,3,5,4)

Denote such a graph n-CYL-P

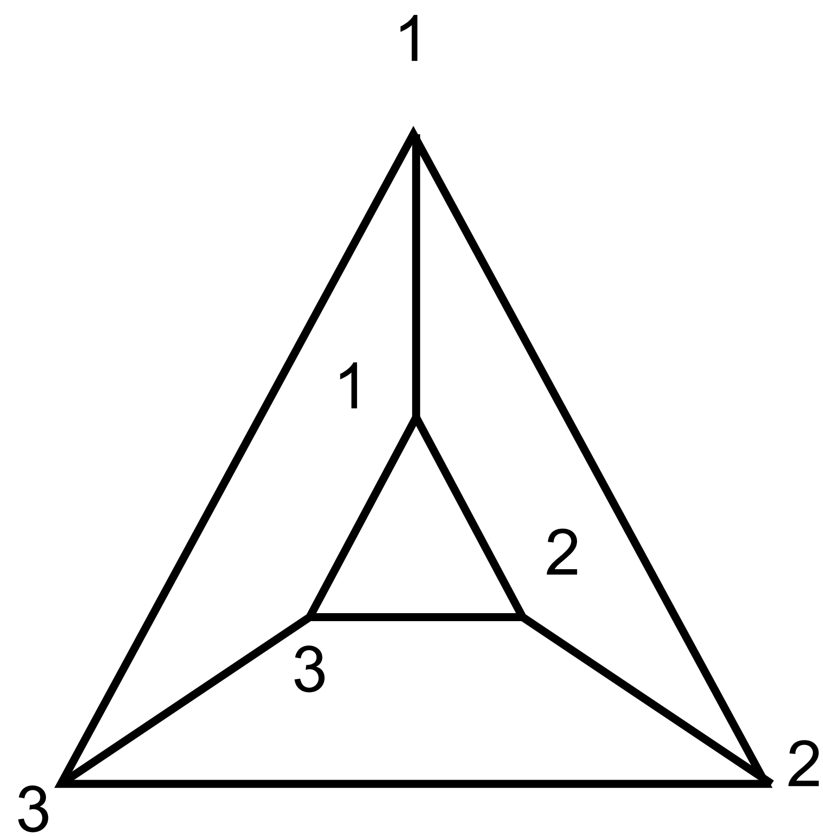
Example. 4-CYL-{1234}



4-CYL-{1243}

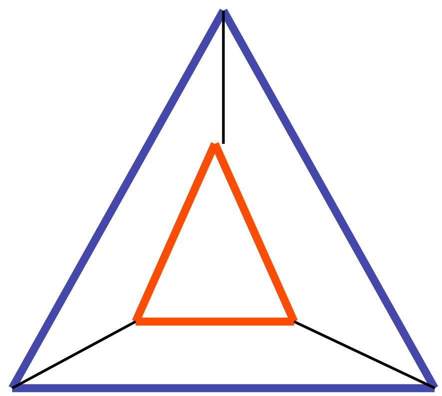


3-CYL-{123}



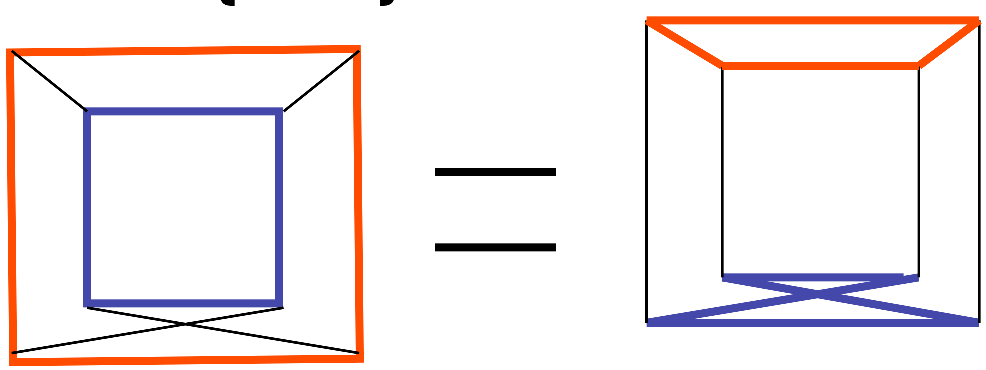
Cycle Partitions

3-CYL{123}



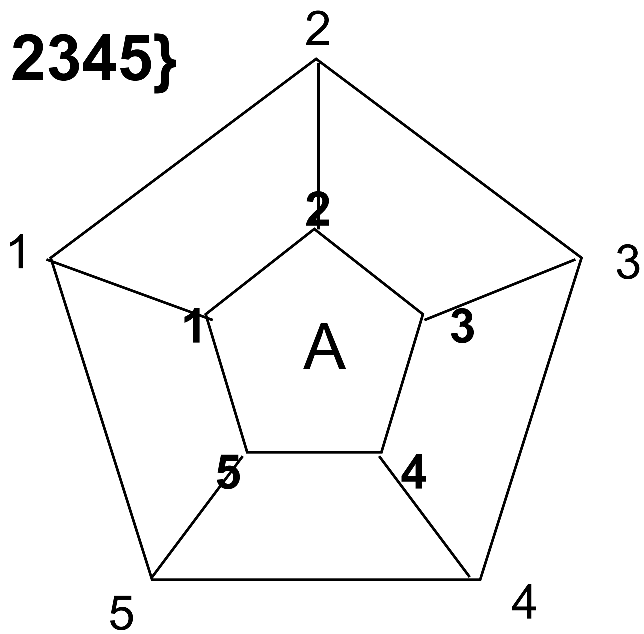
1 cycle-partition
so 4 fixed points

4-CYL{1243}

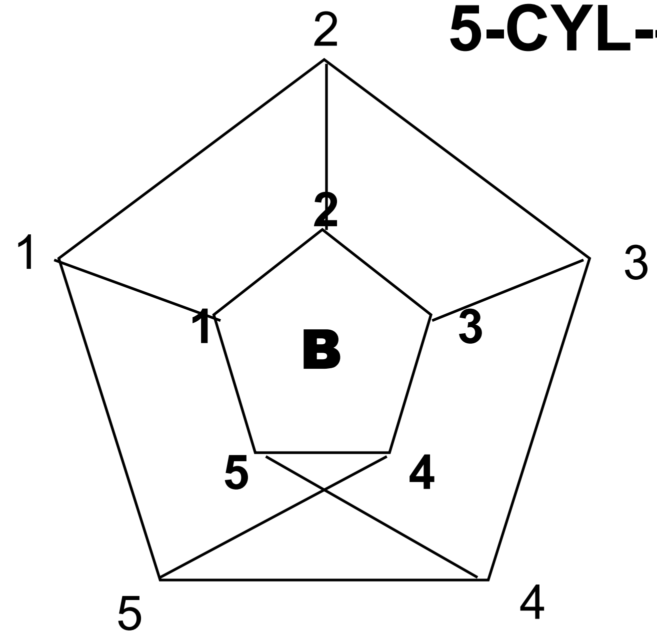


2 cycle-partitions
so 6 fixed points

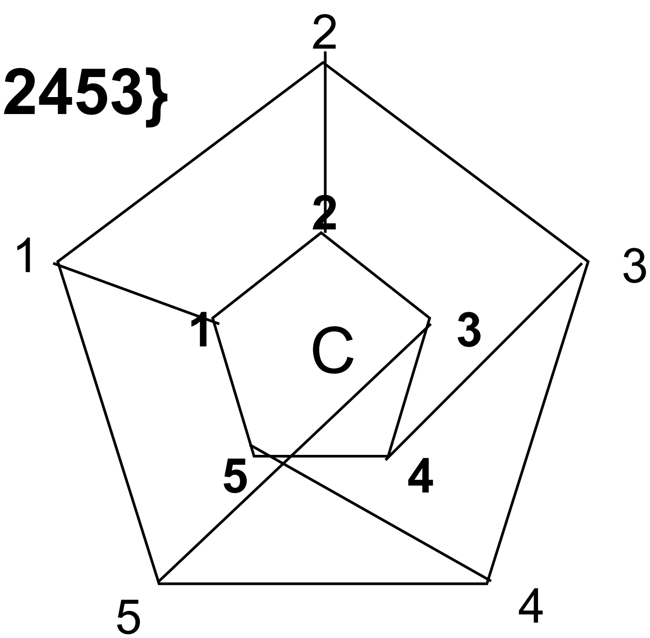
5-CYL-{12345}



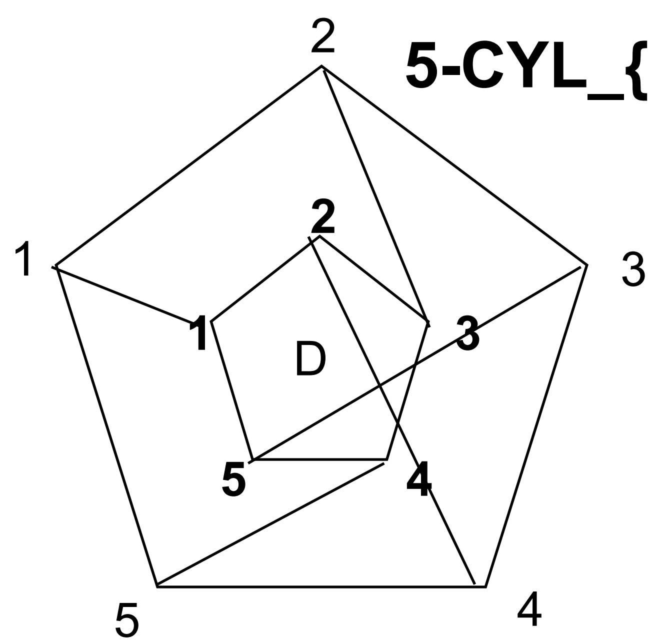
5-CYL-{12354}



5-CYL-{12453}

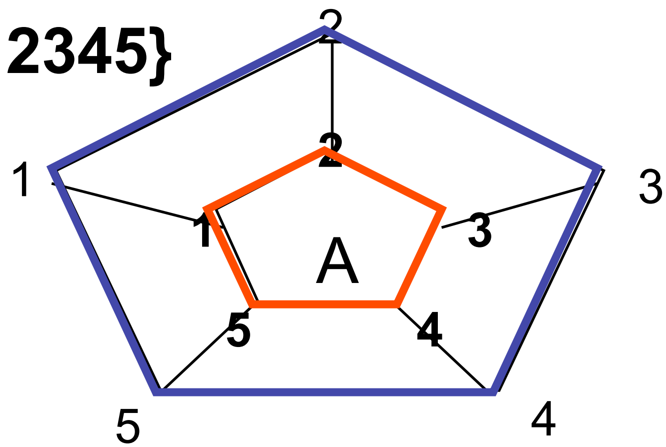


5-CYL_{13524}

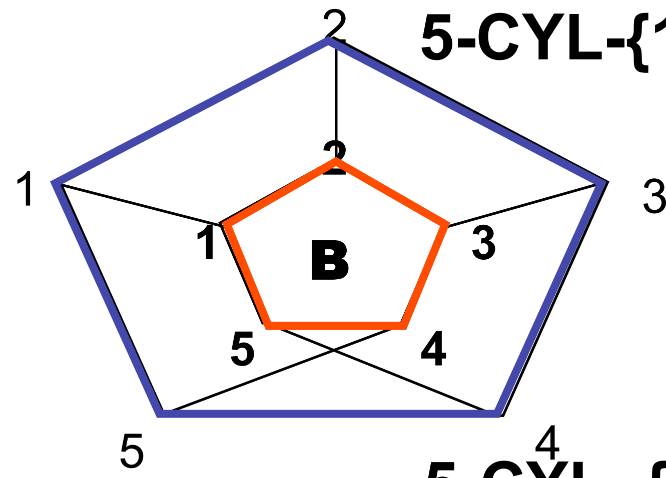


5/5 Cycle Partitions

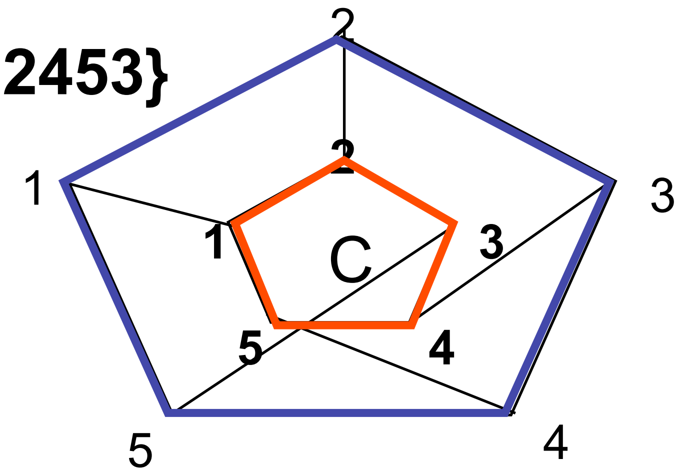
5-CYL-{12345}



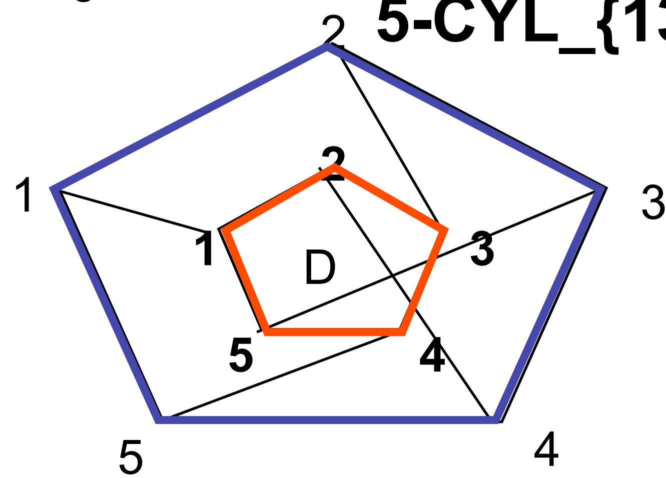
5-CYL-{12354}



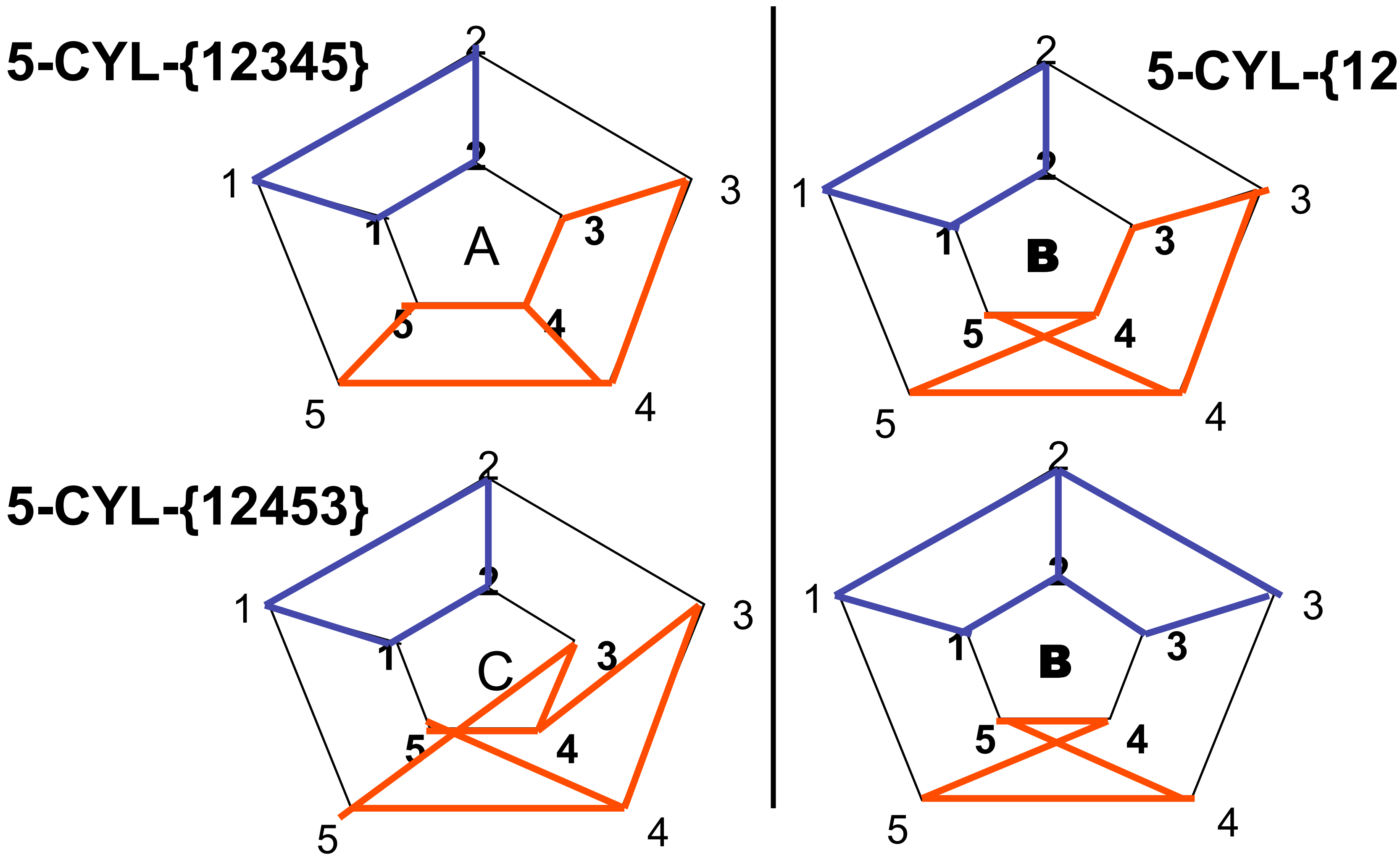
5-CYL-{12453}



5-CYL_{13524}



4/6 Cycle Partitions



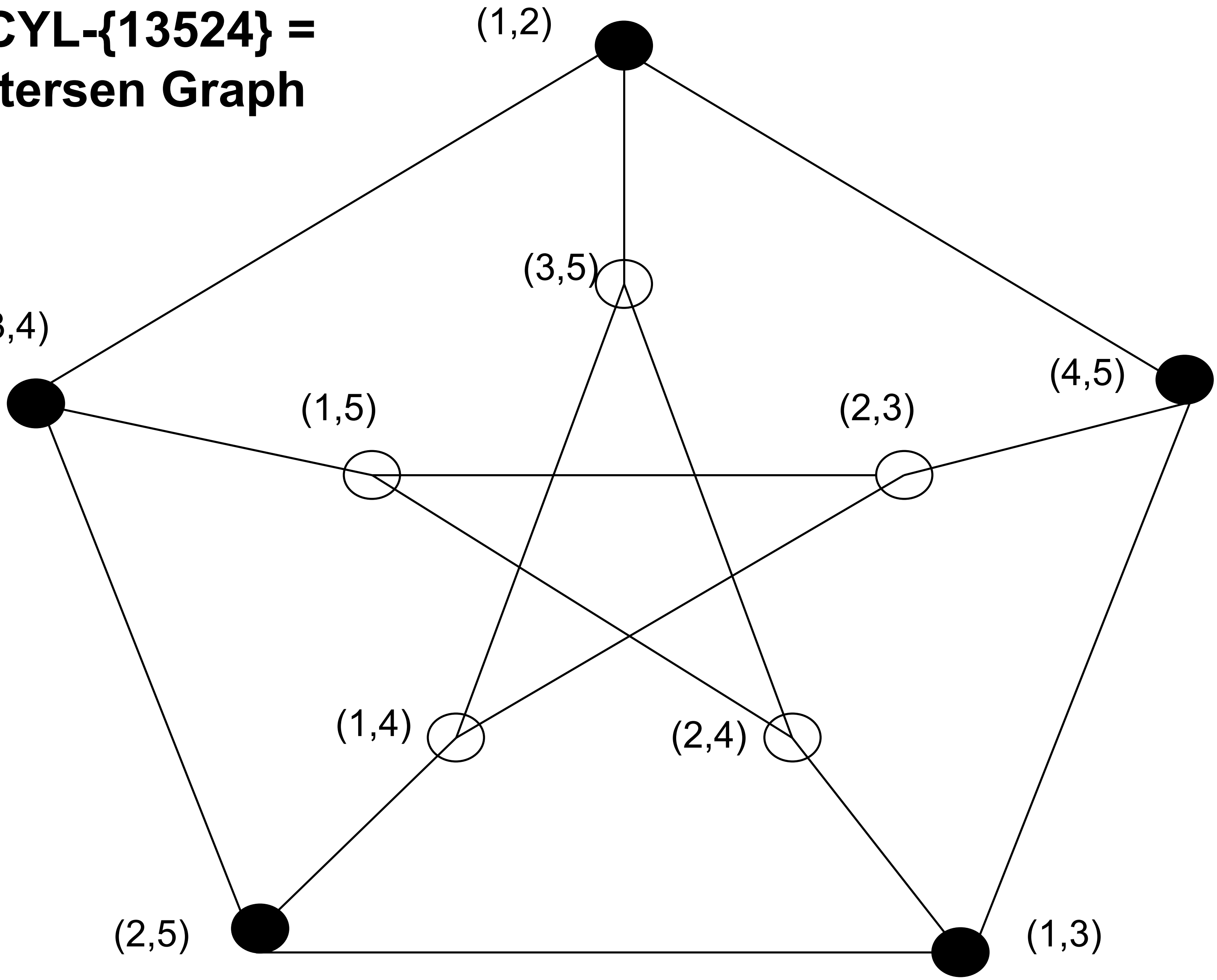
Fixed Points of the Minority Game on Cubic Graphs.

- Suppose we have a state where every vertex is fixed (0 or 1) through time.
- A vertex (playing wlog 0) must have at least two 1-neighbours. Each of these must have at least two 0-neighbours i.e. one additional to our original one, and so on.
- Thus every vertex must be in an alternating circuit of 0 and 1 vertices.

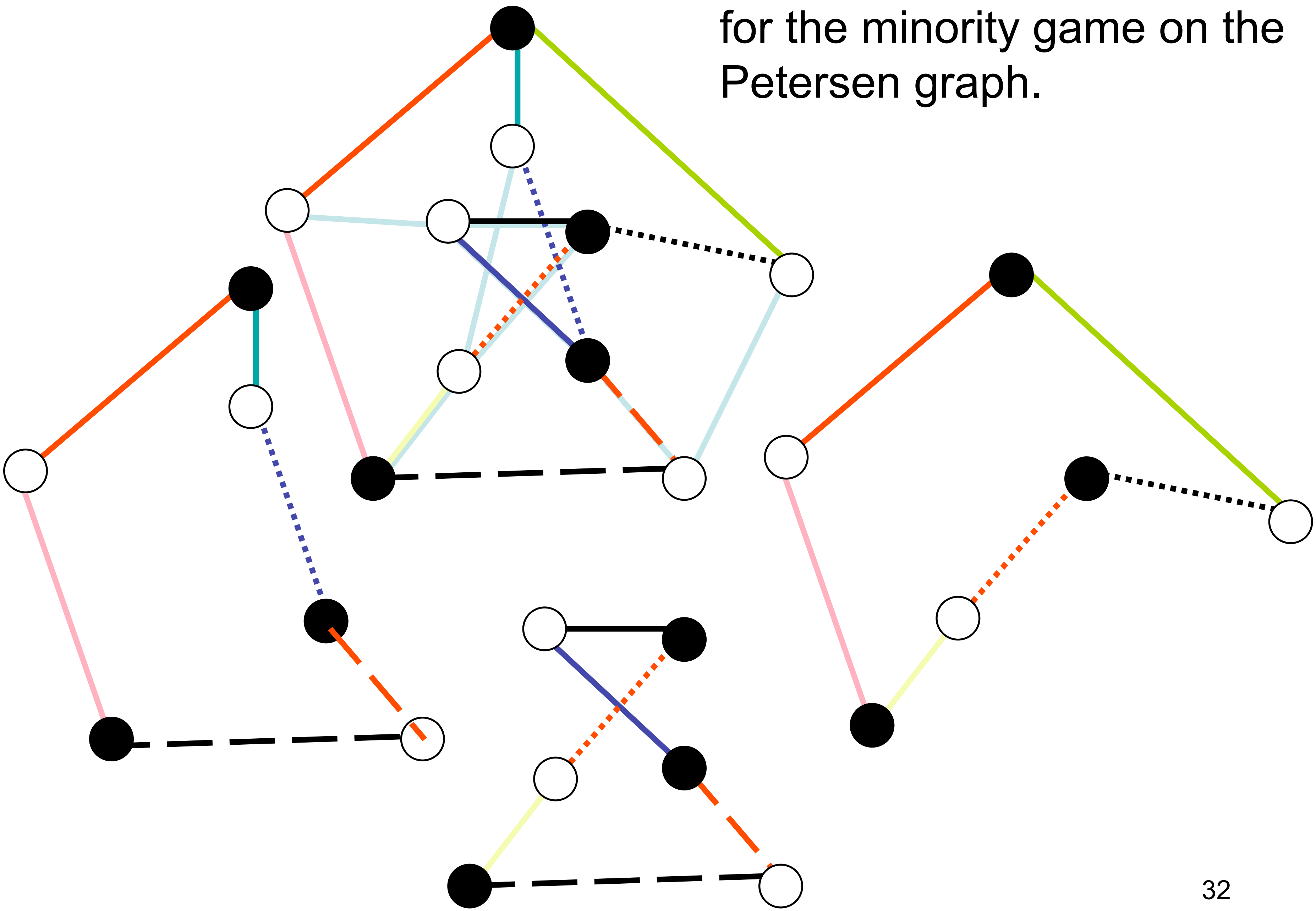
Fixed Points of the Minority Game on Cubic Graphs.

- Much more difficult to find appropriate sets of cycle (which must be of even length), because they are not necessarily disjoint.
- Just give an example; for Petersen graph.

5-CYL-{13524} =
Petersen Graph



“Unique” form of fixed point for the minority game on the Petersen graph.



Mixed Populations

- We now suppose that there is a mixture of M majority and m minority players in the network. Each player consistently employs a majority or a minority strategy.
- Players only observe 0/1 status of neighbours, not maj/min status.
- We shall again suppose that they are repeatedly challenged by randomly assigned initial states.

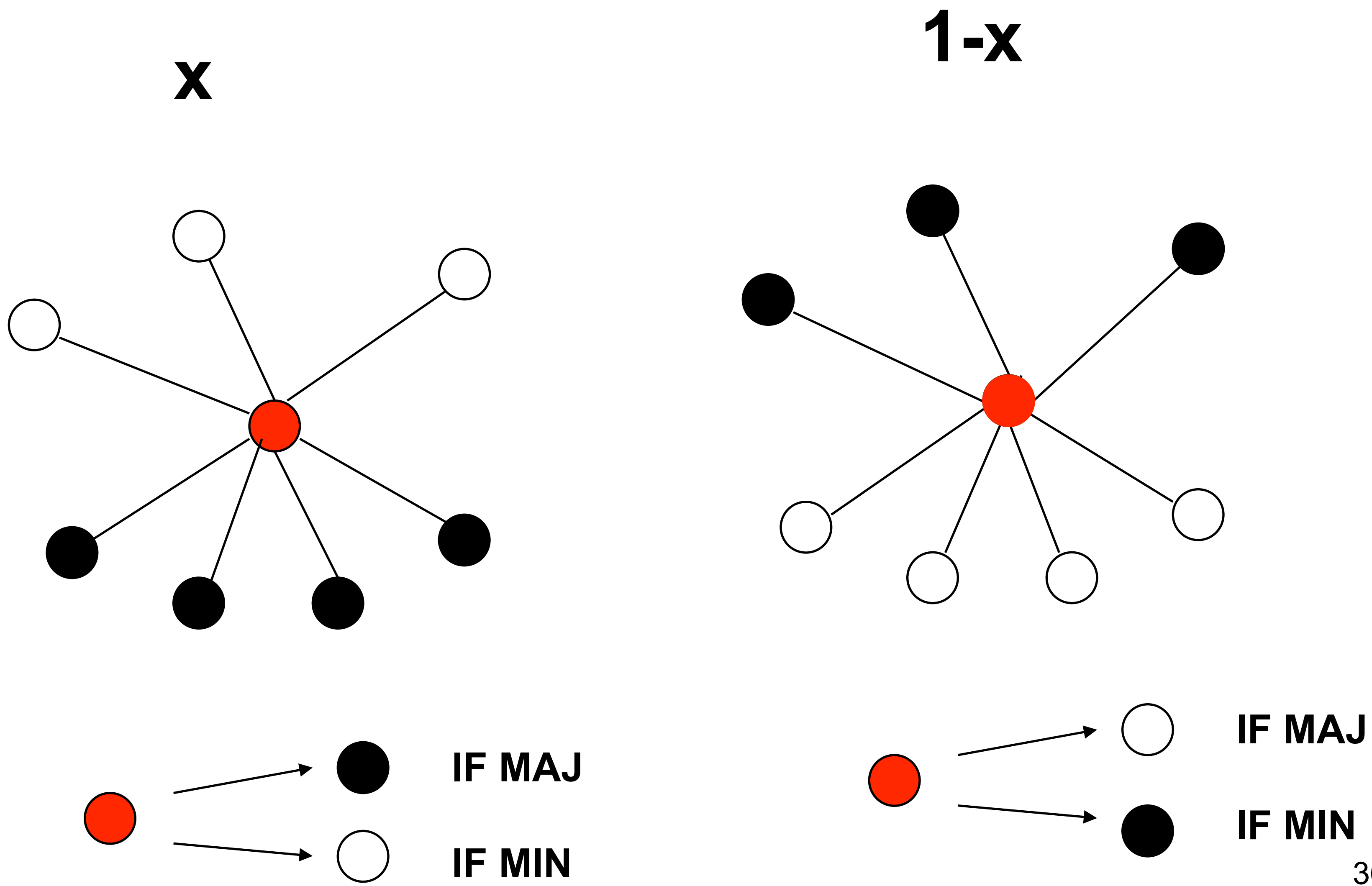
Notation

- We differentiate between the player vector **z** specifying which of majority player (indicated by 1) and minority player (by 0) each vertex (fixed in each game),
  and
- The state vector **x** specifying which of the possible plays (0 or 1) each vertex is using (changing through time).

$F(x)=\mathbf{1}-F(\mathbf{1}-x)$

- We denote by $F(\mathbf{z},\mathbf{x}_t)=x_{t+1}$ the successor to state vector $\mathbf{x}_t$ when the player vector is **z.** For ease we shall usually just use $F(\mathbf{x}_t)$
- Now we consider the class of games on networks where $F(\mathbf{x})= \mathbf{1} – F(\mathbf{1} – \mathbf{x})$ which includes all cases where each individual plays as a majority, or as a minority, player (though not only these).

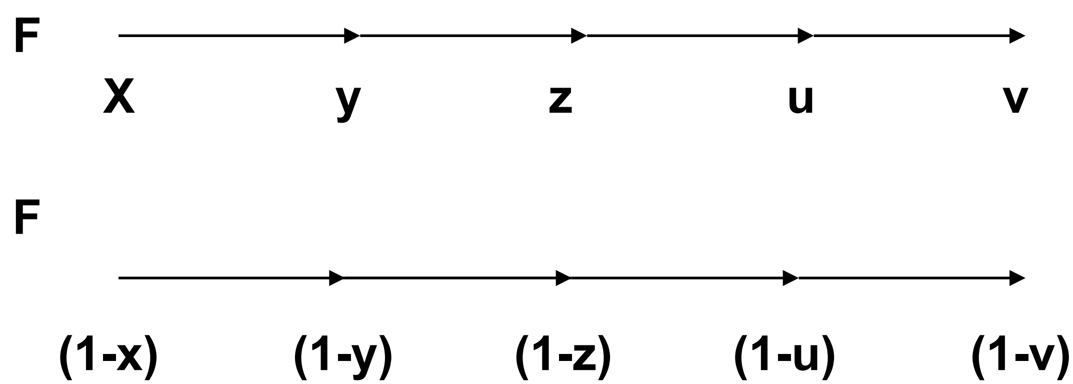
$$F(\mathbf{X}) = 1 - F(1 - \mathbf{X})$$



$F_r(\mathbf{x}) = \mathbf{1}\text{-}F_r(\mathbf{1}\text{-}\mathbf{x})$

- We write $F_r(\mathbf{x})$ for the rth iterate of F on **x**.
- Straightforwardly (by induction) we obtain

$$F_r(\mathbf{x})=\mathbf{1}\text{-}F_r(\mathbf{1}\text{-}\mathbf{x})$$

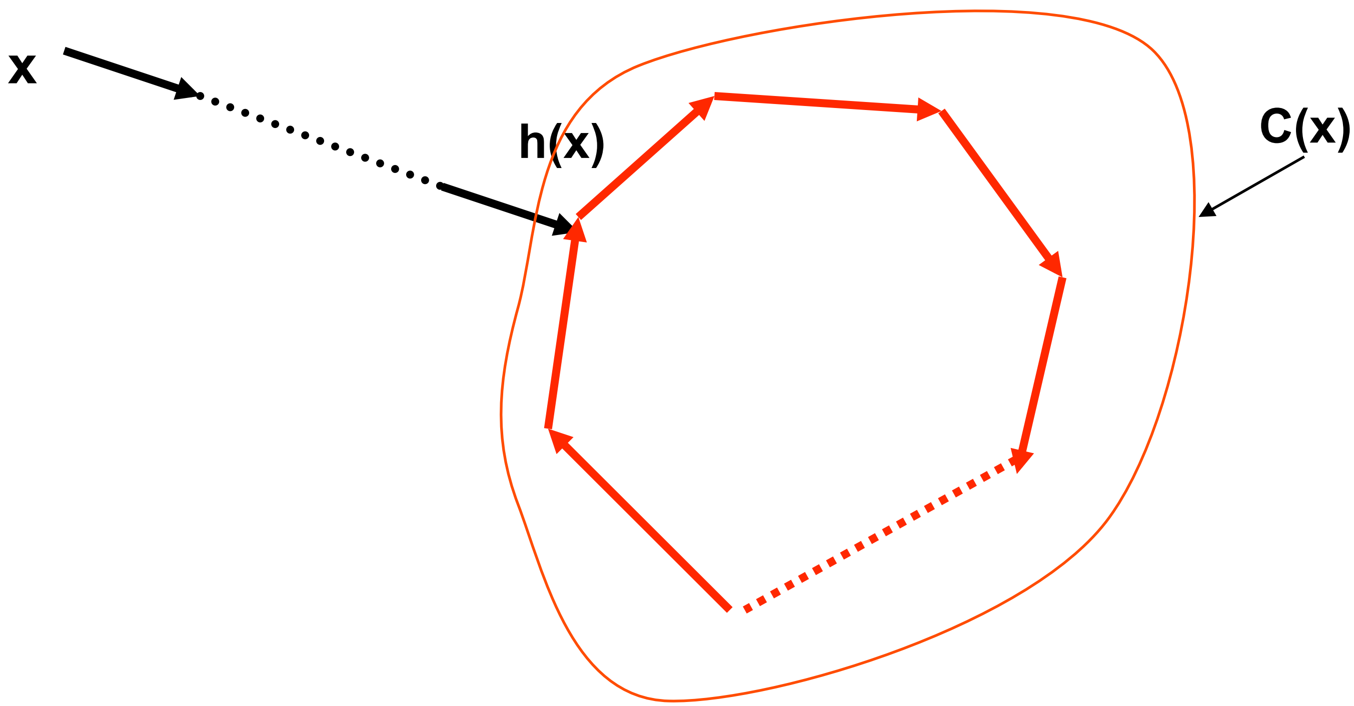


then

Trajectories/Limits for F

- All trajectories reach limit cycles (since finite networks only here). For any state **x** denote C(**x**) the set of states in the limit cycle reached from **x**, and let h(**x**) be the first element of the sequence $\{F_r(\mathbf{x})\}$ which belongs to C(**x**).

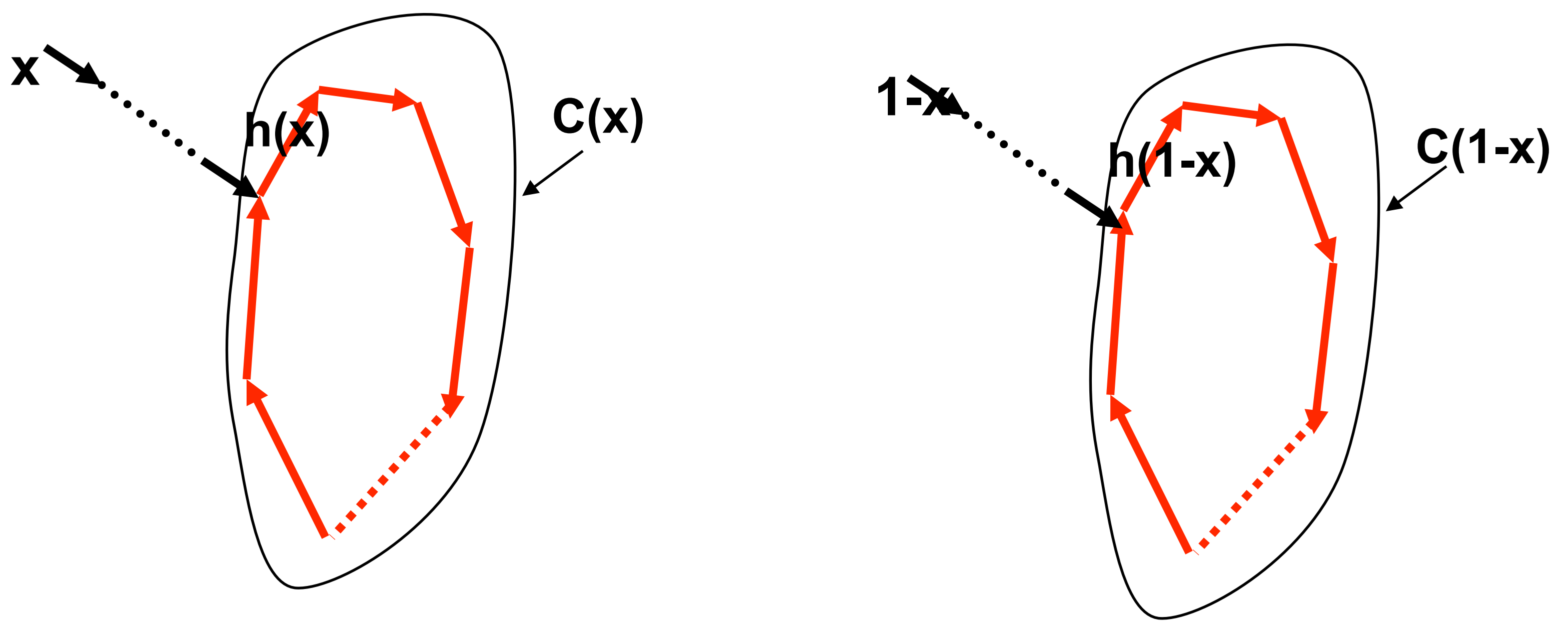
Trajectories/Limits for F



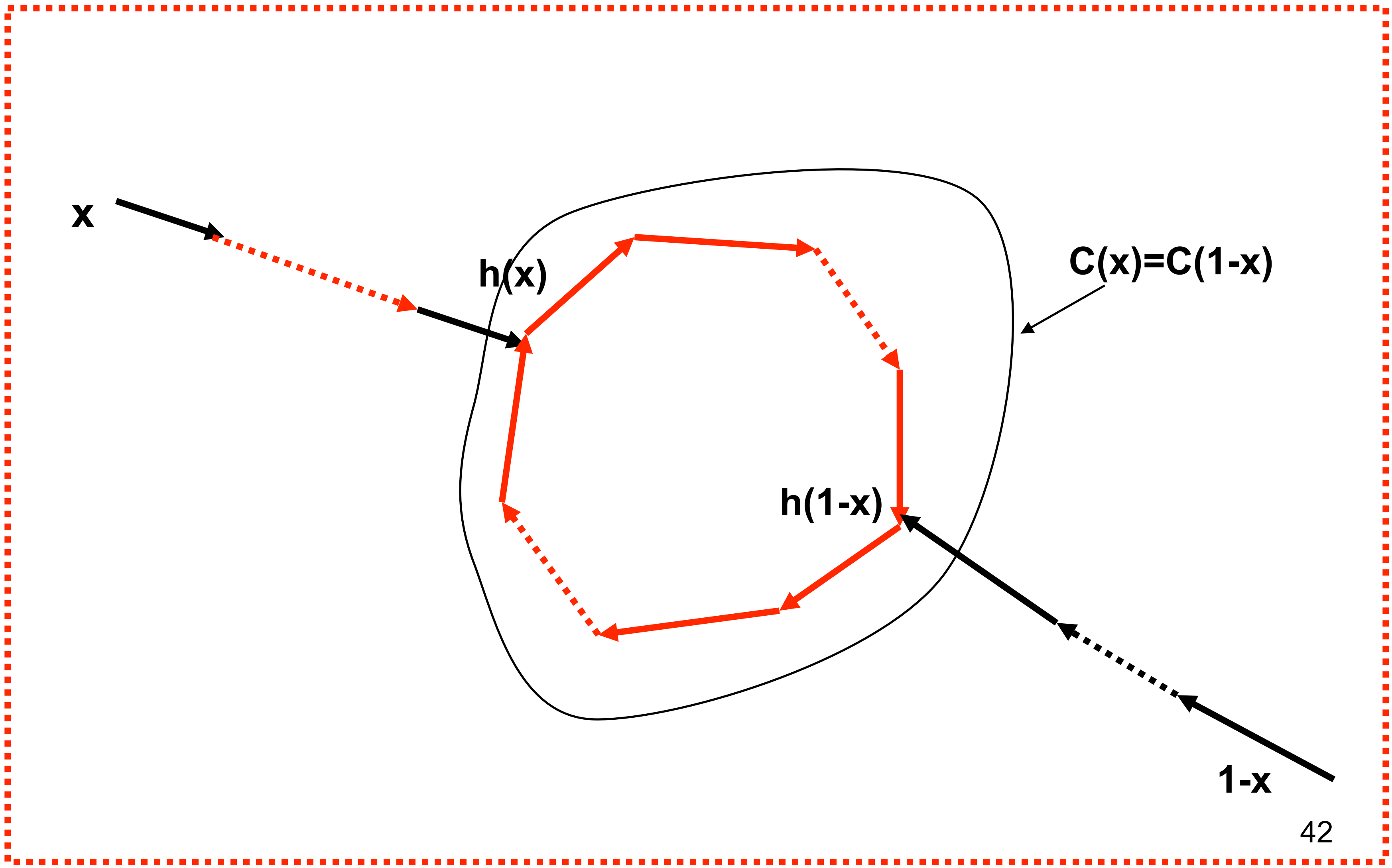
Trajectories/Limits for F

- Clearly h(**1**-**x**)=**1**-h(**x**)
- Suppose **y**=h(**1**-**x**) ∈C(**x**) then for some s F_s(**y**)=h(**x**)=**1**-**y**, and thus F_s(**1**-**y**)=h(**1**-**x**)=**y** and so the cycle length is 2s (i.e. even).
- If x and 1-x lead to distinct cycles then these will be of the same length, which may be odd or even.

Trajectories/Limits for C(x)≠C(1-x)



Trajs/Limits for C(x)=C(1-x)



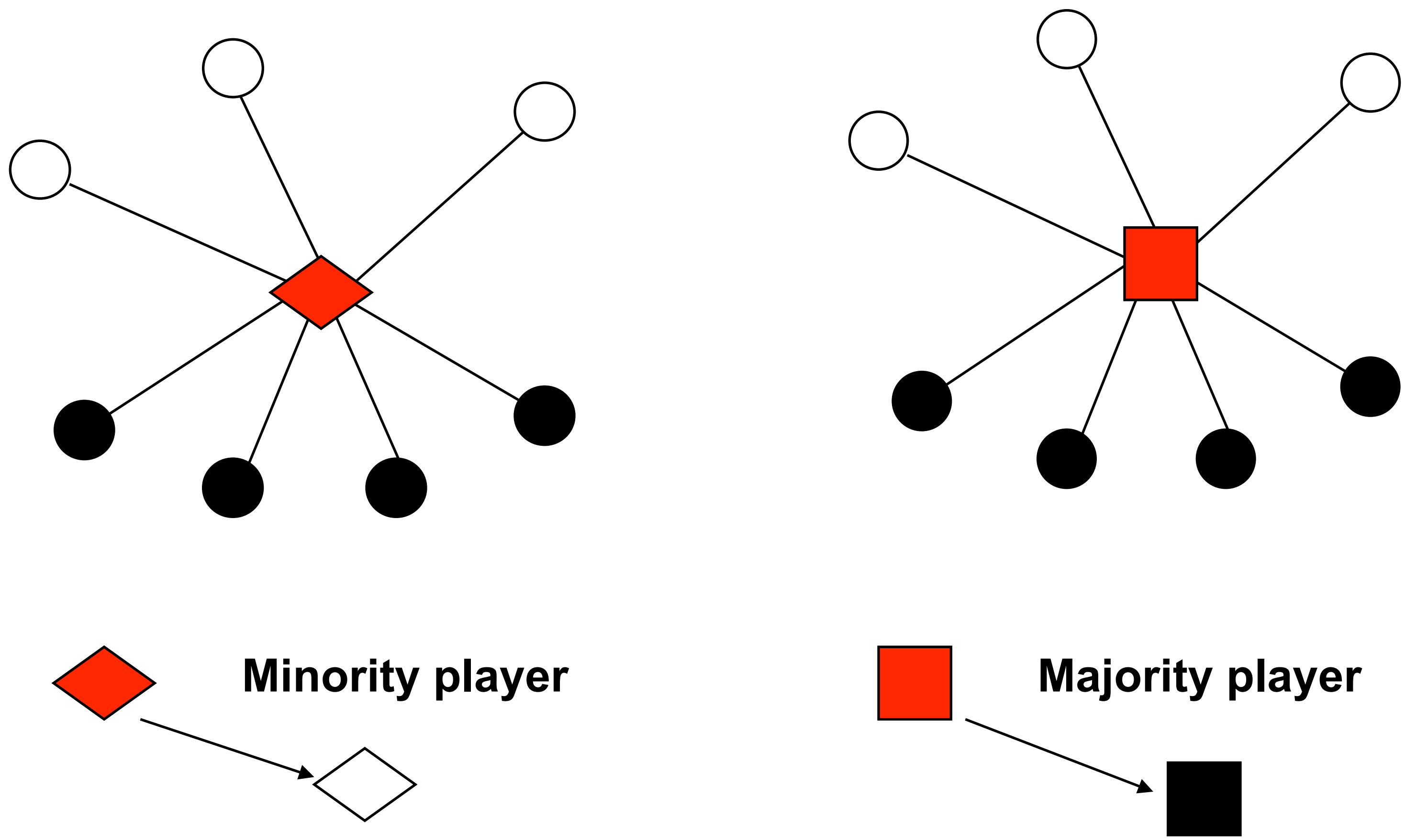
Balance

- Every vertex in C(x) U C(1-x) will have states 0 and 1 with equal frequencies.

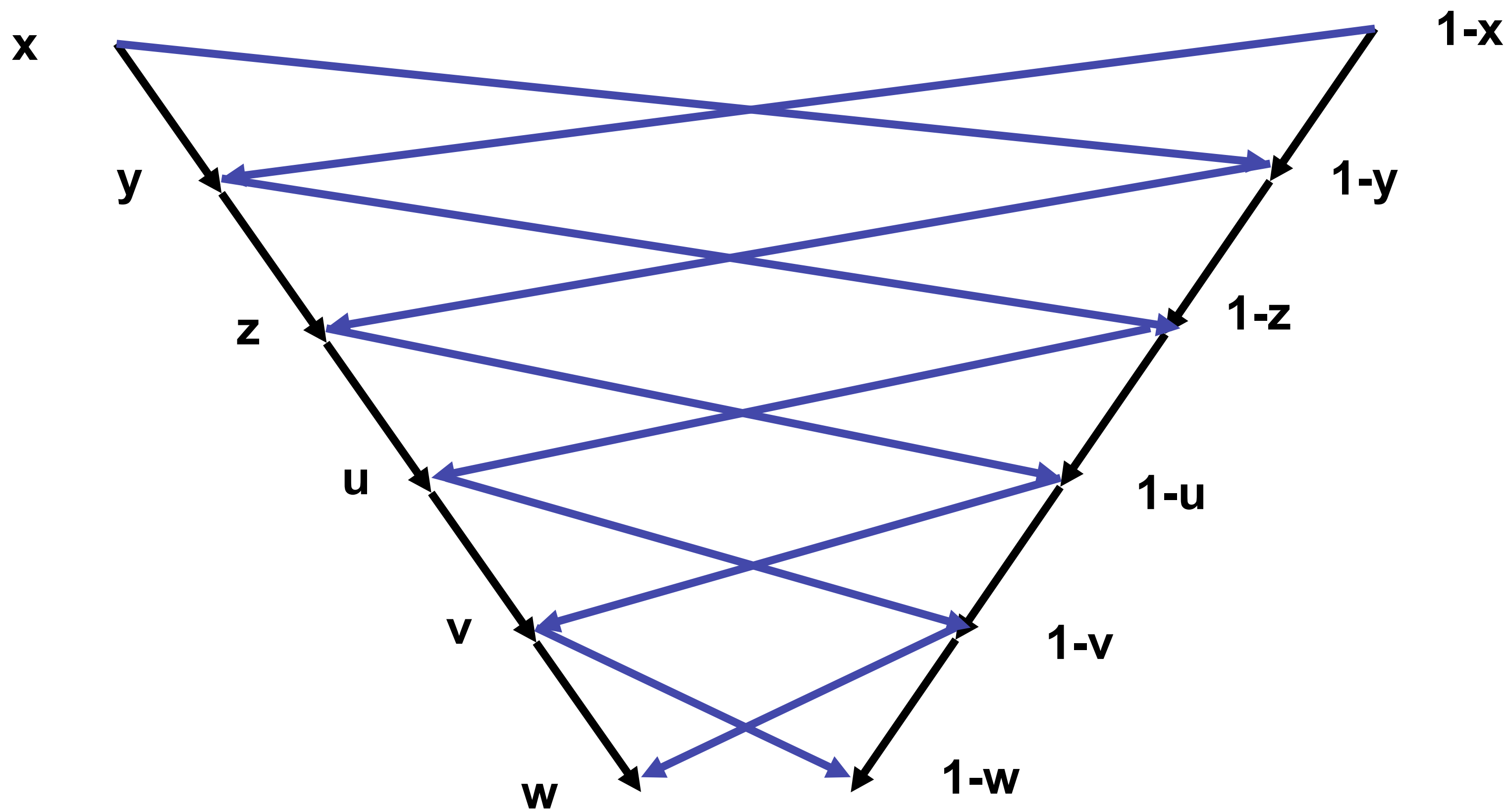
Reversible Games

- Suppose now that we consider a reversal of the assignment of majority and minority players to the vertices, so consider F(**1**-**z**,**x**) which when **z** is clear we write as F*(**x**).
- We term such a game reversible (it is more general than those we consider here) if F*(**x**)=**1**-F(**x**)=F(**1**-**x**).

$F^*(\mathbf{x}) = \mathbf{1} - F(\mathbf{x}) = F(\mathbf{1} - \mathbf{x})$



Dynamics of Reversible Games

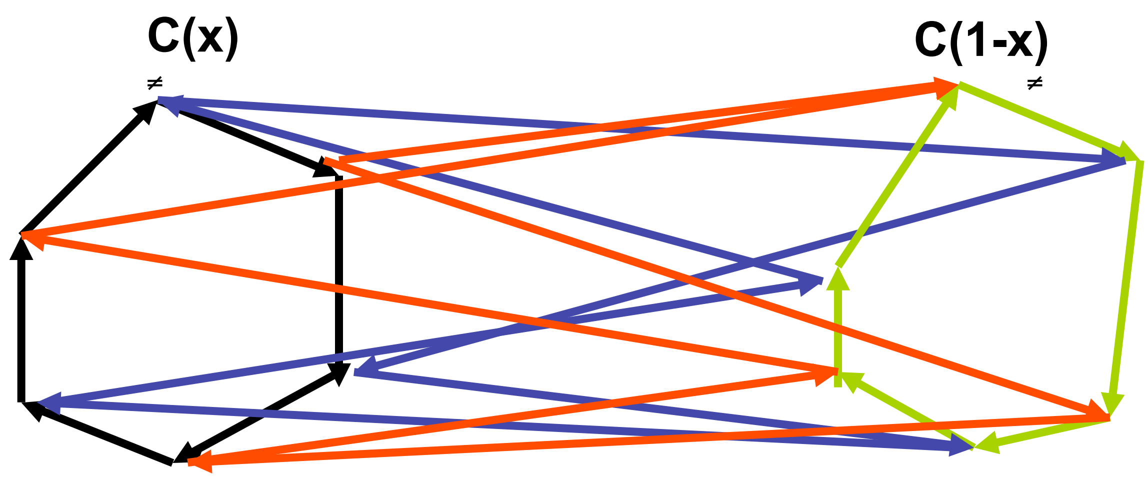


F

$F^*(\mathbf{x})=F(1-\mathbf{x})$

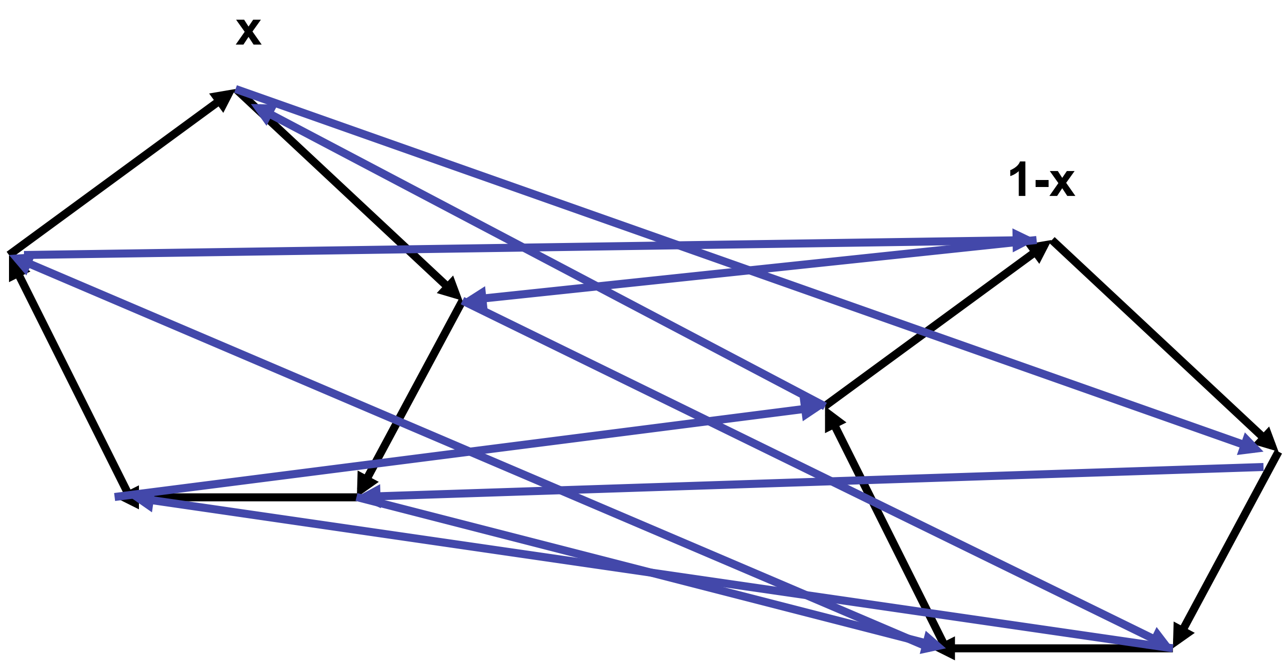
F*

F: C(x) ≠ C(1-x), m(even)-cycles



Two m-cycles arise for F*, C*(x) ne C*(1-x)

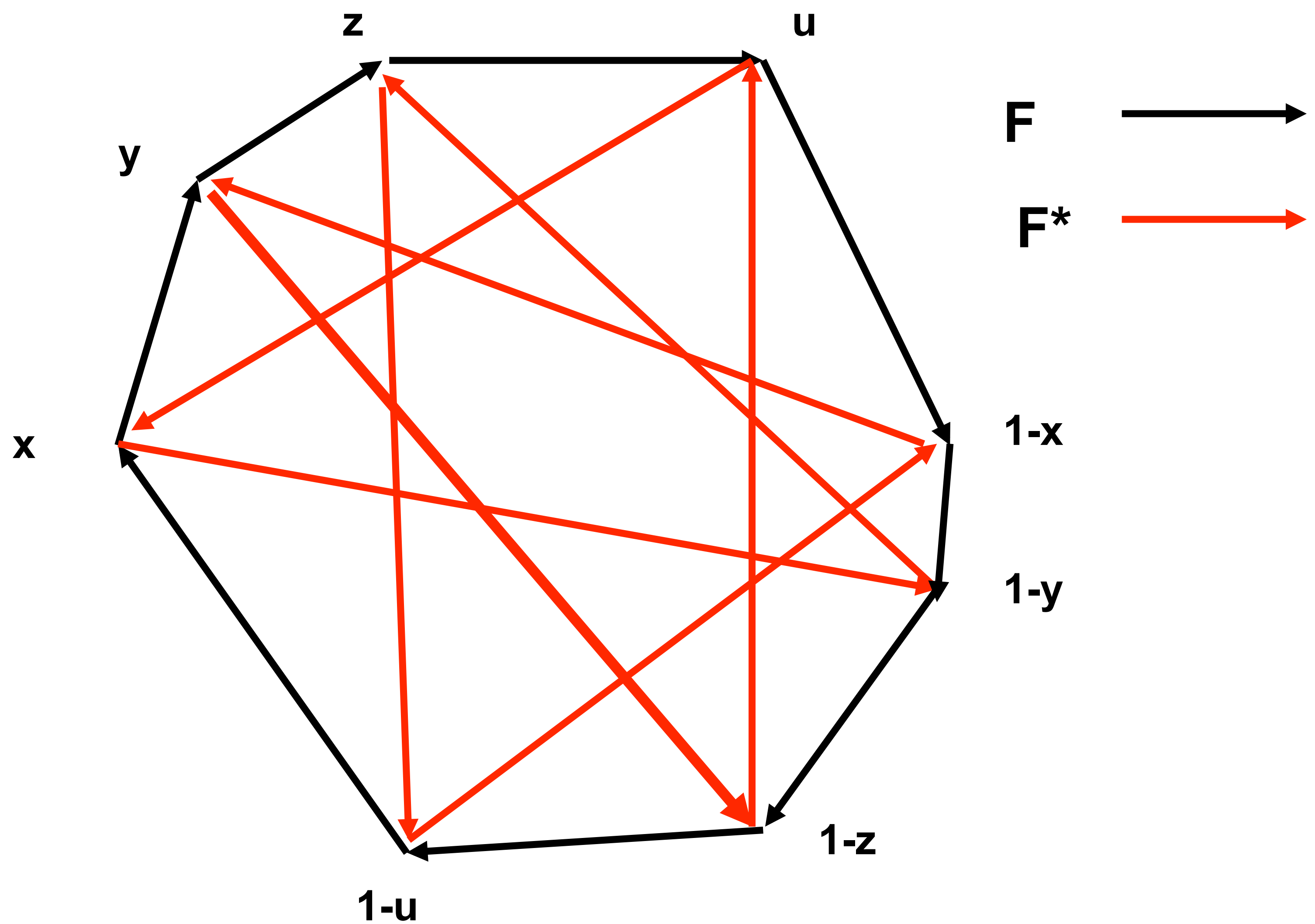
F: C(x) ≠ C(1-x), m(odd)-cycles



One 2m-cycle arises for F*, C*(x)=C*(1-x)

F ⟶ **F*** ⟶

F: C(x)=C(1-x), 2m(m even)-cycle

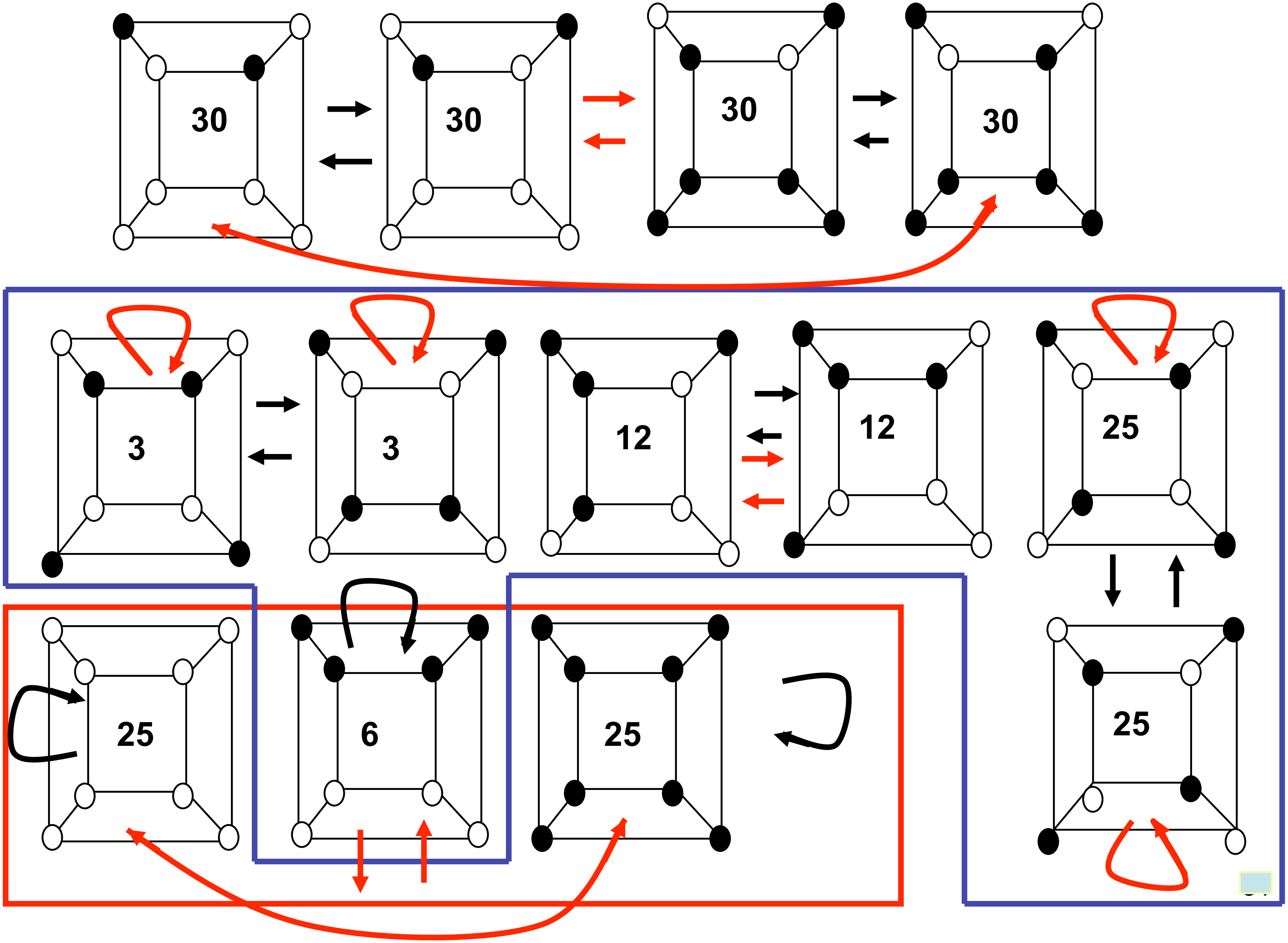


One 2m-cycle arises for F*, C*(x)=C*(1-x)

$U_x\{C(x)\}=U_x\{C^*(x)\}$

- We have demonstrated that the set of states which belong to a limit cycle for F precisely match those of F* which belong to a limit cycle.
- The basins of attraction of $C(x)$ U $C(1-x)$ and of $C^*(x)$ U $C^*(1-x)$ are identical.

LIMIT CYCLES FOR MAJORITY & MINORITY GAME



Mixed Populations (Recap)

- There is a mixture of M majority and m minority vertices in the network. Each vertex consistently employs a majority or a minority strategy.
- Players only observe 0/1 status of neighbours, not maj/min status.
- They are repeatedly challenged by randomly assigned initial states.

Basins of Attaction and Limit Cycles Lengths for 3-cube.

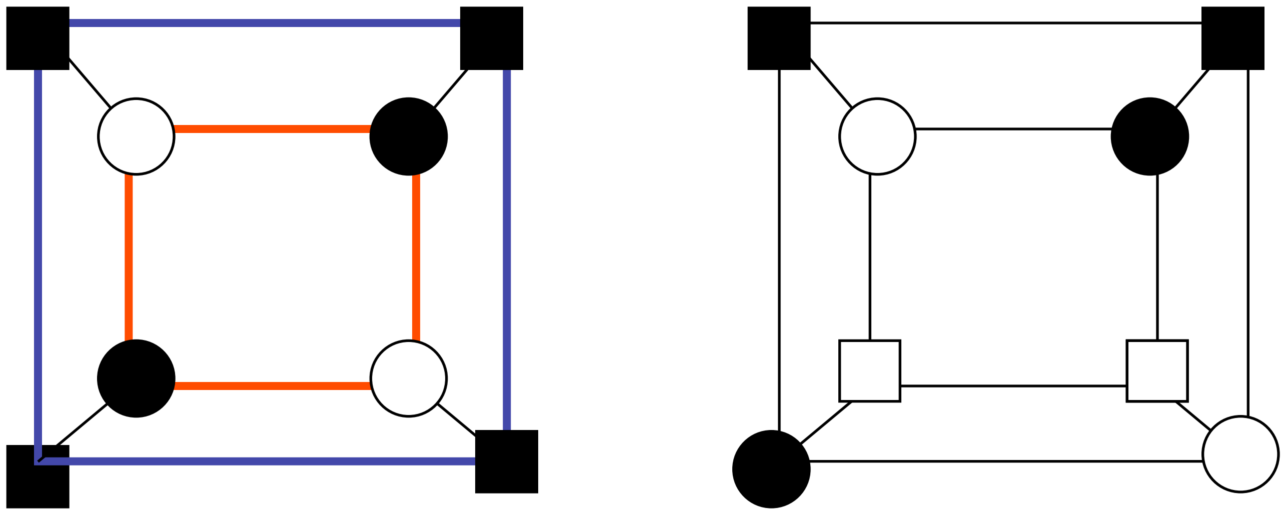
TABLE 1. Number of cycles of various lengths according to number of majority players in mixed game on 3-cube

			Cycle	Length		
		1	2	4	8	Totals
M	0	56	200	0	0	256
A	1	1,024	1,024	0	0	2048
J	2	2,048	2,048	3,072	0	7,168
	3	3,072	3,072	8,192	0	14,436
I	4	144	816	10,816	6,144	17,920
N	5	3,072	3,072	8,192	0	14,436
D	6	2,048	2,048	3,072	0	7,168
'	7	1,024	1,024	0	0	2048
S	8	56	200	0	0	256
	Totals	12,544	13,504	33,334	6,144	65,536

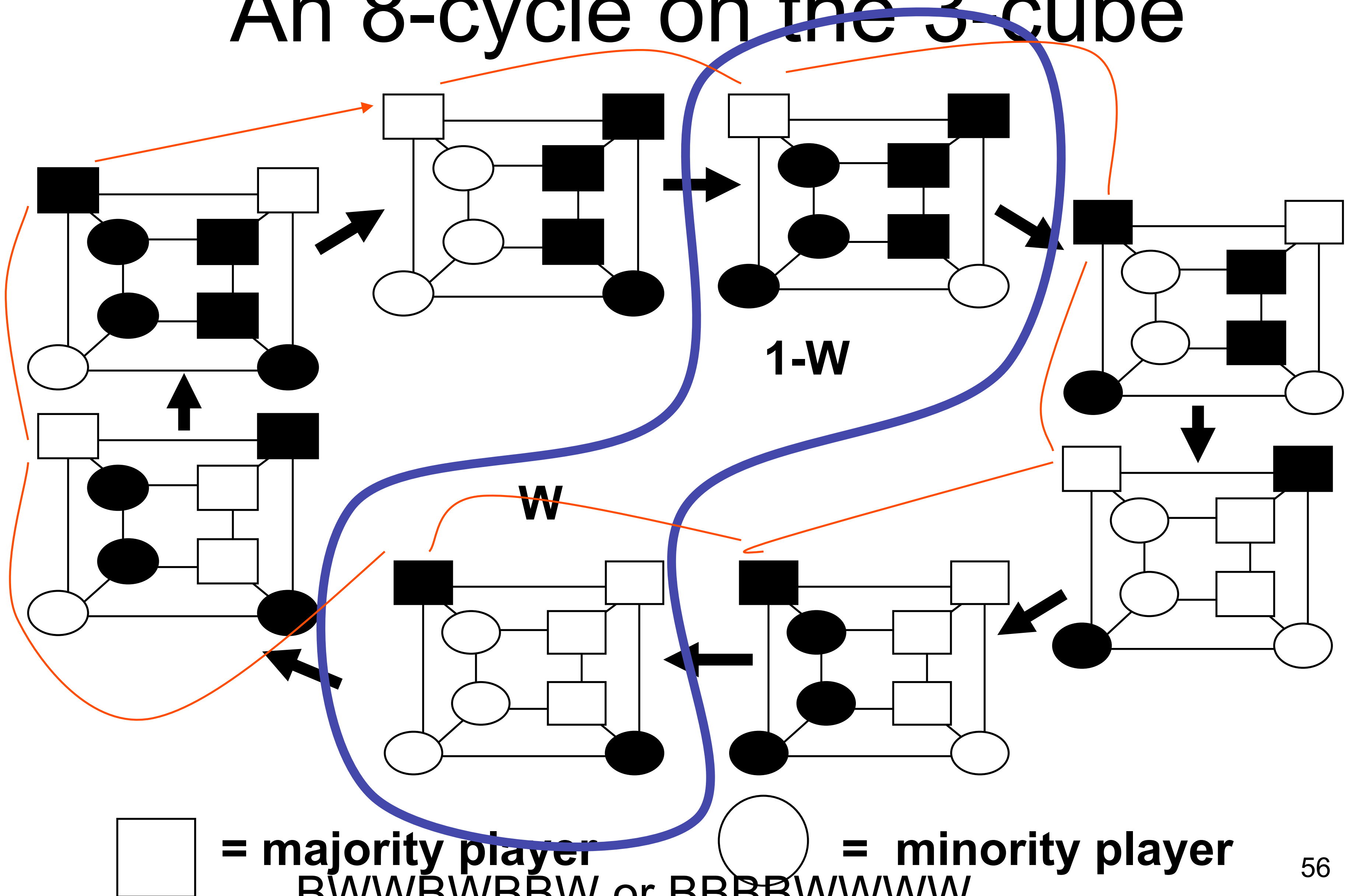
Mixed Game on 3-cube

- Symmetric. e.g. M=5,m=3 and M=3,m=5 give similar distributions.
- Longer cycles when closer match of M and m.
- Not restricted to 1-cycle and 2-cycles.
- Cycle lengths are powers of two.

Two Fixed Points for 4 Maj/4 Min on 3-cube.



An 8-cycle on the 3-cube



= majority player = minority player

BWWBWBBW or BBBBWWWW

Programme

- For a graph with n vertices there are 2^n state vectors and 2^n strategy vectors.
- We can thus explore the dynamics exhaustively for n small.
- It would be preferable to generate the permutationally distinct state/strategy configurations via Polya’s theorem. There are 1996 configurations on the three cube vis-à-vis 2^{16}=65,536.

TABLE 1. Number of cycles of various lengths according to number of majority players in mixed game on Petersen graph

			Cycle	Length		
		1	2	3	4	5
M	0	100	924	0	0	0
A	1	1,560	7,880	0	800	0
J	2	420	20,100	0	19,080	0
	3	3840	26,240	0	55,360	0
I	4	3820	7,260	28,080	135,560	0
N	5	2,400	7968	32,880	72,960	960
D	6	3,680	7,400	720	135,560	0
'	7	17,720	12,360	2640	55,360	0
S	8	20,100	420	0	19,080	0
	9	6,200	3,240	0	800	0
	10	464	560	0	0	0
	Σ	60,304	94,352	64,320	494,560	960

Flow for Majority Game on the Petersen Graph.

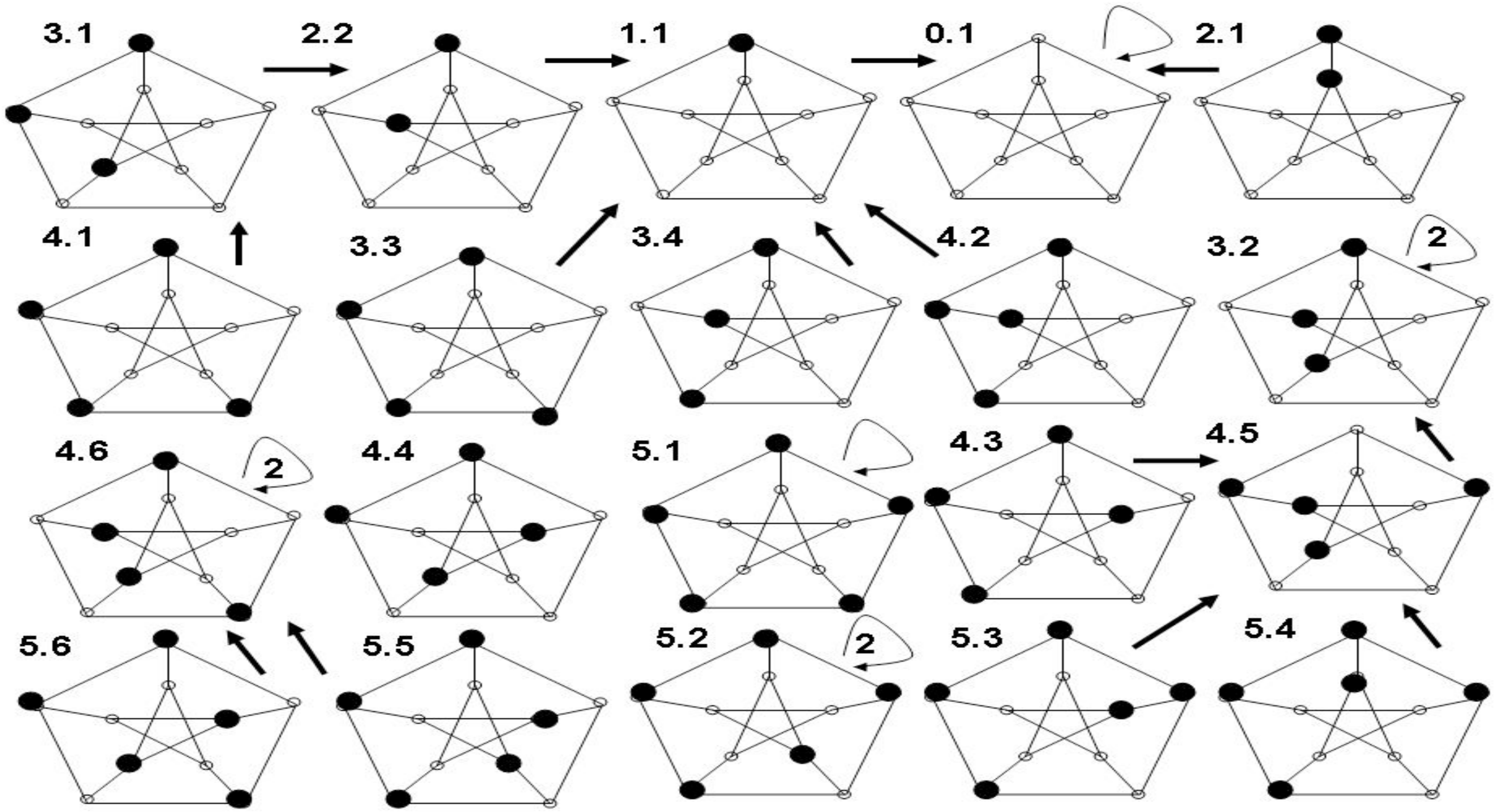


TABLE 2. Number of cycles of various lengths according to number of majority players in mixed game on Petersen graph

				Cycle	Length		
		6	8	10	16	32	Total
M	0	0	0	0	0	0	1,024
A	1	0	0	0	0	0	10,240
J	2	6,480	0	0	0	0	46,080
	3	6,720	30,720	0	0	0	122,880
I	4	36,720	2,160	0	1,440	0	215,040
N	5	85,680	19,680	960	0	34,560	258,048
D	6	64,080	2,160	0	1,440	0	215,040
'	7	4,080	30,720	0	0	0	122,880
S	8	6,480	0	0	0	0	46,080
	9	0	0	0	0	0	10,240
	10	0	0	0	0	0	1,024
	Σ	210,240	85,440	960	2,880	34,560	2^{20}

General Features for Reversible Games

- n div 4 then symmetric over M/m (follows from earlier discussion)
- Longest cycles for M=m
- More fixed points for M>m
- More 2-cycles for m>M

Dynamics on the Graphs

- Can we detect features of the graphs which tell us what limits to expect?
- Apart from 3-CYL{123} there are no triangles so the usual clustering measures will be 0.
- We might expect the diameter, the number of symmetries, average distance to have influence.

Dynamics on the Graphs

TABLE 1

Linked Polygon	I	II	III	IV	V	VI	VII
MD	1	3/2	11/8	17/10	87/50	81/50	3/2
OAG	12	48	16	20	4	4	120
NC	6	4	7	16	21	15	10
LC	8	8	16	28	38	24	32
NOC	1	0	1	4	6	4	2
LOC	3	0	3	9	19	9	5

Properties of the Linked Polygonal Graphs for Triangles, Rectangles and Pentagons. Key. I={1,2,3}, II={1,2,3,4}, III={1,2,4,3}, IV={1,2,3,4,5}, V={1,2,3,5,4}, VI={1,2,4,5,3}, VII={1,3,5,2,4}, MD=Mean Distance between Nodes, OAG=Order of Automorphism Group, NC=Number of Cycle Lengths, LC=Longest Cycle Length, NOC=Number of Odd (≥ 3) Cycle Lengths, LOC=Longest Odd Cycle Length

Payoff Matrix

- A Majority player receives 1 for each neighbour whose state matches his own, 0 for any that contrast.
- A Minority player receives 1 for each neighbour whose state contrasts with his own, 0 for any that match.

Payoffs

- Suppose that the players are fixed but their position in the graph and state in the various contests it will engage in are randomly assigned (equal probs etc).
- We can then ask what payoffs will vertices receive. In fact we ask what payoffs will majority/minority vertices receive. Normalised so that random plays would average 1.

Reversible Games

- Define $P_F(k)$=average (over all starting states) payoff per player when there are k majority players.
- $P_F(k)+P_{F^*}(k)=P_F(n-k)+P_F(n-k)$=constant.
- $P_{F,M}(k)$=average payoff to majority players when k majority players, similarly $P_{F,m}(k)$.
- $P_{F,M}(k)+P_{F,m}(n-k)$=constant.

Payoff for the 3-cube

- For the three cube $P_{FM}(k)=1.5$ i.e. just what a random player receives.
- In fact for any bipartite graph this is true.

Payoff to maj/min v number maj.

- Petersen = 5-cylinder{13524}

Numb.Maj.	0	1	2	3	4	5	6	7	8	9	10
Maj Payoff	*****	0.391	0.864	0.903	1.016	1.000	1.121	1.028	1.001	1.339	1.297
Min Payoff	0.703	0.661	0.999	0.972	0.879	1.000	0.984	1.097	1.136	1.609	*****

Nash Equilibria for number of Maj/Min Players
NB. Payoffs scaled so random play gives 1.000
Rather than 1.500.

Nash Equilibria

Maj's->	0	1	2	3	4	5	6	7	8	9	10
123											
1243											
12345											
12354											
12453											
13524											
PT											
L10											

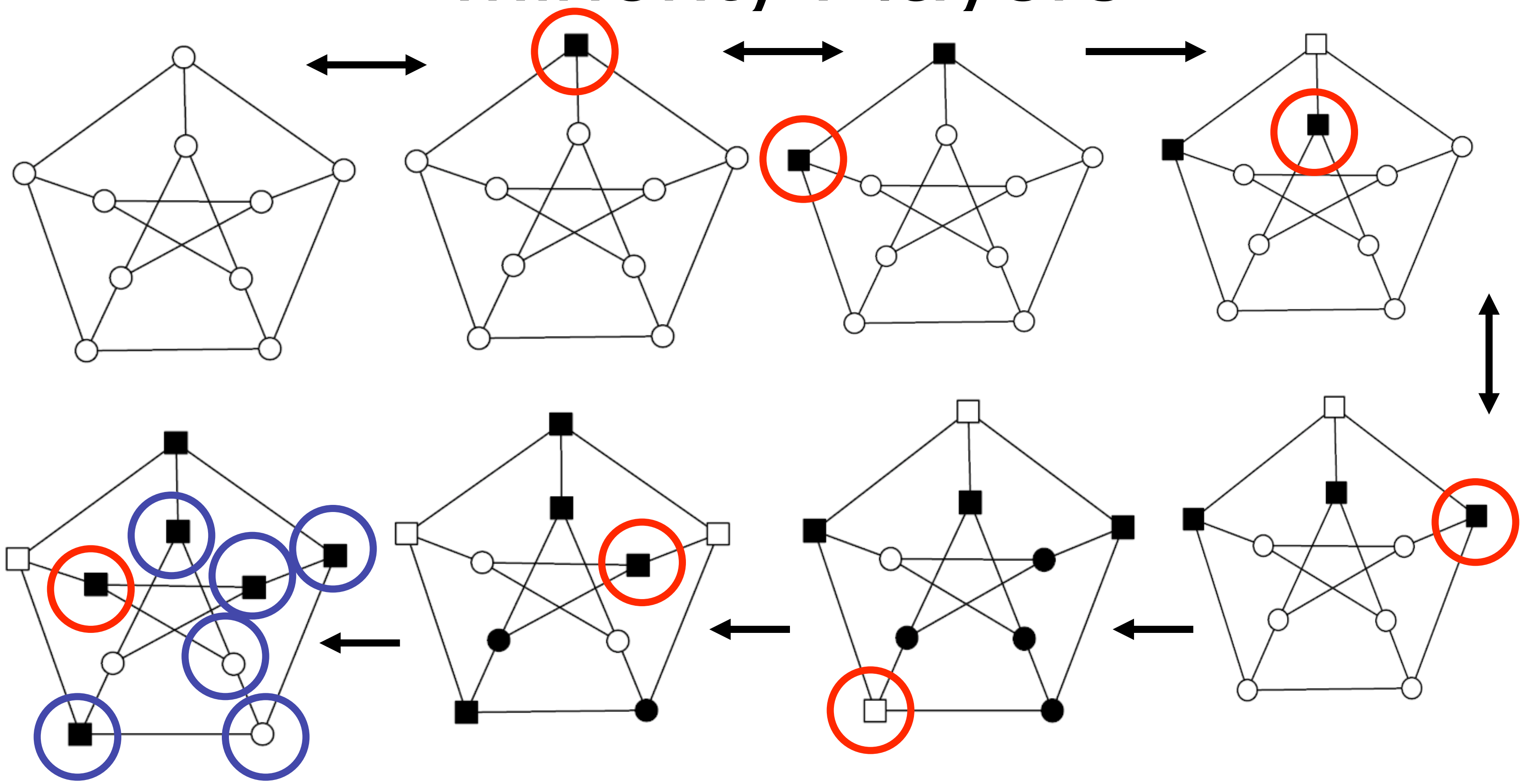
Nash Equilibria

- All-Maj never strict Nash equilibrium.
- All-Min is strict Nash equilibrium.

Fixed Vertex Position

- If the players have fixed positions then the above analysis is not appropriate. Nonetheless some similar results must apply. The following is a simple illustration of the possible behaviour.

Invasion of All Majority Players by Minority Players



Invader ○ Neutral ⟷ Directed ⟵ Invadable ○

Conclusions

- Long cycles may result (longest so far 58)
- Balance of Maj and Min leads to longer cycles, and smaller basin of attractions for fixed points.
- Difficult to link properties of the graph and the dynamics.
- Dynamics of the full game, i.e. where Maj and Min are also changeable is complex.

Future Work

- Cycle partitions.
- More states.
- Higher degrees.
- Larger networks.
- Evolving networks.
- Less myopia.

Thank you for your attention!